International Journal on Recent and Innovation Trends in Computing and Communication
ISSN: 2321-8169 Volume: 11 Issue: 11
***Article Received:** 10 September 2023 **Revised:** 20 October 2023 **Accepted:** 30 October 2023*

Triple Even Star Decomposition of Complete Bipartite Graphs

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Abstract
Let G be a finite, connected, undirected graph without loops or multiple edges.A decomposition $\{G_2, G_4, \ldots, G_{2k}\}$ of G is said to be an even star decomposition if each G_i is a star and $|E(G_i)| = i$ for all $i = 2, 4, \ldots, 2k$. A graph G is said to have Triple Even Star Decomposition (TESD) if G can be decomposed into $3k$ stars $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$. In this paper, we characterize Triple Even Star Decomposition of complete bipartite graphs $K_{m,n}$ when $m = 2$ and $m = 3$.

Keywords: Complete bipartite graph, Star, Decomposition.
2010 Mathematics Subject Classification: 05C51, 05C30.

1. Introduction

Let $G = (V, E)$ be a simple, connected graph with p vertices and q edges. A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$. The graph $K_{1,r}$ is called a star and is denoted by S_r. A star with centre i and end vertices $1', 2', \ldots, n'$ is denoted by $(i; 1', 2', \ldots, n')$. Terms not defined here are used in the sense of [5].

A decomposition of a graph G is a family of edge-disjoint subgraphs $\{G_1, G_2, \ldots, G_k\}$ such that $E(G)=E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. A decomposition $\{G_1, G_2, \ldots, G_k\}$ for all $k \in N$ is said to be a *Continuous Monotonic Decomposition (CMD)* if each G_i is connected and $|E(G_i)| = i$ for all $i \in N$. The concept of CMD was introduced by Joseph and Gnanadhas[6].

A decomposition $\{G_1, G_2, \ldots, G_n\}$ of G said to be an *Arithmetic Decomposition (AD)* if $|E(G_i)| = a + (i - 1)d$ for all $i = 1, 2, \ldots, n$ and $a, d \in Z^+$. Clearly $q = \frac{n}{2}[2a + (n - 1)d]$. If $a = 1$ and $d = 1$, then AD is a CMD. If $a = 1$ and $d = 2$ in AD, then it is called an *Arithmetic Odd Decomposition (AOD)*. The concept of Arithmetic Odd Decomposition(AOD) was introduced by Merly and Gnanadhas[1].

The concept of Double Arithmetic Odd Decomposition (DAOD) was introduced by Shali and Asha[8]. The concept of Even Star Decompositionof Complete Bipartite graphs was introduced by Merly and Goldy[2].

In this paper, we give characterization for $K_{m,n}$ when $m = 2$ and $m = 3$ which admits Triple Even Star Decomposition (TESD).

2 Triple Even Star Decomposition of $K_{2,n}$

In this section, we give characterization for K_2,n to be Triple Even Star Decomposable.

Definition 2.1. A graph G is said to admit *Even Star Decomposition (ESD)* if G can be decomposed into k stars $\{S_2, S_4, \ldots, S_{2k}\} \forall\ k \in N$.

Theorem 2.2. [2] Any graph G admits Even Decomposition $\{G_2, G_4, G_6, \ldots, G_{2n}\}$, where $G_{2i} = (V_{2i}, E_{2i})$ and $|E(G_{2i})| = 2i$, for all $(i = 1, 2, 3, \ldots, n)$ if and only if $q = n(n+1)$ for some $n \in Z^+$.

Theorem 2.3. [3] Any graph G admits Double Even Decomposition $(2G_2, 2G_4, \ldots, 2G_{2n})$ where $G_{2i} = (V_{2i}, E_{2i})$ and $|E(G_{2i})| = 2i$, for all. $(i = 1, 2, \ldots, n)$ if and only if $q = 2n(n+1)$ for some $n \in Z^+$.

Theorem 2.4.[3] Let n be a positive integer with $n \geq 2$. Then $K_{2,n}$ admits Double Even Star Decomposition $\{2S_2, 2S_4, \ldots, 2S_{2k}\}$ [2k-decomposition]with $k = s$ iff $n = s^2 + s$; $s \in N$.

Definition 2.5. A graph G is said to have *Triple Even Star Decomposition(TESD)* if G can be decomposed into $3k$ stars $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$. It is called as a 3k-decomposition of G.Clearly, number of edges = $3k(k+1)$.

Theorem2.6. Any graph G admits Triple Even Decomposition $(3G_2, 3G_4, \ldots, 3G_{2n})$where $G_{2i} = (V_{2i}, E_{2i})$ and $|E(G_{2i})| = 2i$, for all $(i = 1, 2, \ldots, n)$ if and only if
$q = 3n(n+1)$ for some $n \in Z^+$.

Proof. Suppose $q = 3n(n+1)$ for each $n \in Z^+$.Apply induction on n.
The result is obvious when $n = 1$ and $n = 2$. Suppose the result is true when $n = k$.
Let G be any connected graph with $q = 3k(k+1)$. Then G can be decomposedinto $(3G_2, 3G_4, 3G_6, \ldots, 3G_{2k})$.
We prove that the result is true for $n = k+1$.
Let G' be any connected graph with 3(k+1)[k+1+1] edges.
We prove that G' admits $(3G_2, 3G_4 \ldots\ldots, 3G_{2k}, 3G_{2(k+1)})$
Thus $q(G') = 3[k(k+1) + 2(k+1)] = 3k(k+1) + 6(k+1)$.
Let G^* and G^{**} be two subgraphs of G with $3k(k+1)$ and $6(k+1)$ edgesrespectively.
By induction hypothesis G^* can be decomposed into 3k subgraphs$(3G_2, 3G_4, \ldots, 3G_{2k})$.
Therefore G can be decomposed into $(3G_2, 3G_4, \ldots 3G_{2k})$.
Now $|E(G^{**})| = 6(k+1) = 3(k+1) + 3(k+1)$ which can be decomposed intotwo subgraphs G^{***} and G^{****} each of $3(k+1)$ edges.
Hence G admits Triple Even Decomposition.
Conversely, Suppose G admits TED $(3G_2, 3G_4, 3G_6, \ldots, 3G_{2k})$.
Then $q(G) = 3n(n+1), n \in Z^+$.

Now, let us decompose $K_{m,n}$ when $m = 2$.

Theorem 2.7. Let n be a positive integer with $n > 2$. Then $K_{2,n}$ admits Triple Even Star Decomposition $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$ [3k-decomposition]with $k = 2s$ or $k = 2s-1$, $s \in N$ iff $n = 3s(2s \pm 1)$; $s \in N$.

Proof. Let $G = K_{2,n}$ with $n \in N$ and $n > 2$. Then $|E(K_{2,n})| = 2n$. Assume that $K_{2,n}$ has a TESD $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$. Clearly $2n = 3k(k+1)$ where k denotes the total number of decompositions.Thus, $2n = 3k(k+1) \Rightarrow n = \frac{3k(k+1)}{2}$. Suppose $k = 2s$, then $\Rightarrow n = \frac{3(2s)(2s+1)}{2} \Rightarrow n = 3s(s+1)$.
Suppose $k = 2s-1$, then $\Rightarrow n = \frac{3(2s-1)(2s)}{2} \Rightarrow n = 3s(2s-1)$.
Therefore $n = 3s(2s \pm 1)$.
Conversely assume that $n = 3s(2s \pm 1)$, $s \in N$. Hence n and s are of same parity. Let $K_{m,n} = (V_1(G), V_2(G))$ where $V_1(G) = \{1, 2, \ldots, m\}$ and $V_2(G) = \{1', 2', \ldots, n'\}$.
2

Consider the matrix (a_{ij}) (1)

With $a_{11} = \begin{cases} 4s \text{ if } n = 3s(2s+1) \\ 4s-2 \text{ if } n = 3s(2s-1) \end{cases}$.

Define $a_{1j} = a_{2j}$,

$a_{2j} = a_{2(j+1)}$,
$a_{1(j+1)} = a_{2(j+1)}$ -2,
$a_{1(j+1)} = a_{1(j+2)}$,
$a_{1(j+2)} = a_{2(j+2)}$,
$a_{2(j+3)} = a_{2(j+2)} - 2$,
$a_{2(j+3)} = a_{1(j+3)}$,
$a_{1(j+3)} = a_{1(j+4)}$,
$a_{2(j+4)} = a_{1(j+4)} - 2$,
$a_{2(j+4)} = a_{2(j+5)}$,
$a_{2(j+5)} = a_{1(j+5)}$; $j = 1, 7, 13, \ldots, (6t-5)$.

The entry $a_{1\overline{6t}} = 2 + 8\left(\frac{s}{2}\right) - t$; t = 1, 2, …,$\frac{s}{2}$ and the consecutive entry,

$a_{1\overline{6t+1}} = a_{1\overline{6t}} - 2$; t = 1, 2, …,$\left(\frac{s}{2} - 1\right)$.

Case(i): $n = 3s(2s \pm 1)$ where n and s are even

Consider the matrix (a_{ij}) as in (1) with order $\begin{cases} 2 \times 3s & \text{if } n = 3s(2s+1) \\ 2 \times (3s-1) & \text{if } n = 3s(2s-1) \end{cases}$

as follows:

Clearly number of entries are $\begin{cases} 6s & \text{if } n = 3s(2s+1) \\ (6s-2) & \text{if } n = 3s(2s-1) \end{cases}$ and also sum of each row is n as well as ends in $\begin{cases} 2 & \text{if } n = 3s(2s+1) \\ 0 & \text{if } n = 3s(2s-1) \end{cases}$.

Thus we have 2s different entries $\begin{cases} 2, 4, 6, \ldots, 4s & \text{if } n = 3s(2s+1) \\ 0, 2, 4, 6, \ldots, 4s-2 & \text{if } n = 3s(2s-1) \end{cases}$ in the matrix (a_{ij}) with each entry repeated thrice.

Now edges incident with i can be decomposed into 3s stars $S_{a_{i1}}, S_{a_{i2}}, \ldots, S_{a_{i(3s)}}$ for $1 \le j \le 3s$ where

$$S_{a_{ij}} = (i; [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 1]', [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 2]', \ldots, [a_{i1} + a_{i2} + \ldots + a_{ij}]'); \boldsymbol{j} = 1, 2, \ldots, 3s.$$

or now edges incident with i can be decomposed into (3s-1) stars $S_{a_{i1}}, S_{a_{i2}}, \ldots, S_{a_{i(3s-1)}}$ if $a_{ij} \ge 0$ for $1 \le j \le (3s-1)$ where

$$S_{a_{ij}} = (i; [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 1]', [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 2]', \ldots, [a_{i1} + a_{i2} + \ldots + a_{ij}]'); \boldsymbol{j} = 1, 2, \ldots, (3s-1)$$

Thus $E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s}) \cup E(S_{4s}) \cup E(S_{4s})$ if $a_{11} = 4s$

$E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s-2}) \cup E(S_{4s-2}) \cup E(S_{4s-2})$ if $a_{11} = 4s-2$.

Hence $K_{2,n}$ admits Triple Even Star Decomposition $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$ with $k = 2s$ when $n = 3s(2s+1)$ (or) $k = 2s - 1$ when $n = 3s(2s-1)$; $s \in N$.

Case(ii): $n = 3s(2s \pm 1)$ where n and s are oddNow, $E(K_{2,n}) = E(K_{2,n-1}) \cup E(K_{1,2})$.

Consider $\boldsymbol{K}_{2,n-1}$. Here (n-1) is even. Consider the matrix (a_{ij}) as in (1) with order $\begin{cases} 2 \times 3s & \text{if } n = 3s(2s+1) \\ 2 \times (3s-1) & \text{if } n = 3s(2s-1) \end{cases}$ as follows:

Clearly number of entries are $\begin{cases} 6s & \text{if } n = 3s(2s+1) \\ (6s-4) & \text{if } n = 3s(2s-1) \end{cases}$ and also sum of each row is n-1 as well as ends in $\begin{cases} 0 & \text{if } n = 3s(2s+1) \\ 2 & \text{if } n = 3s(2s-1) \end{cases}$. But does not contain S_2.

Thus we have $\begin{cases} (2s+1) & \text{if } n = 3s(2s+1) \\ (2s-1) & \text{if } n = 3s(2s-1) \end{cases}$ different entries $\begin{cases} 2, 4, 6, \ldots, 4s & \text{if } n = 3s(2s+1) \\ 0, 2, 4, 6, \ldots, 4s-2 & \text{if } n = 3s(2s-1) \end{cases}$ in two rows. Totally we have $\begin{cases} (2s+1) & \text{if } n = 3s(2s+1) \\ (2s-1) & \text{if } n = 3s(2s-1) \end{cases}$ different entries in the matrix (a_{ij}) with each entry repeated thrice.

Now edges incident with i can be decomposed into 3s stars $S_{a_{i1}}, S_{a_{i2}}, \ldots, S_{a_{i(3s)}}$ if $a_{ij} \geq 0$ for $1 \leq j \leq 3s$ where

$$S_{a_{ij}} = (i; [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 1]', [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 2]', \ldots, [a_{i1} + a_{i2} + \ldots + a_{ij}]'); \; j = 1, 2, \ldots, 3s$$

or now edges incident with i can be decomposed into (3s-2) stars $S_{a_{i1}}, S_{a_{i2}}, \ldots, S_{a_{i(3s-2)}}$ for $1 \leq j \leq (3s-2)$ where

$$S_{a_{ij}} = (i; [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 1], [a_{i1} + a_{i2} + \ldots + a_{i(j-1)} + 2], \ldots, [a_{i1} + a_{i2} + \ldots + a_{ij}]); \; j = 1, 2, \ldots, (3s-2)$$

Thus $E(K_{2,n}) = E(K_{2,n-1}) \cup E(K_{1,2})$.

$$E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s}) \cup E(S_{4s}) \cup E(S_{4s}) \cup E(S_2).$$

$$E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s}) \cup E(S_{4s} \cup E(S_{4s})$$

Hence $K_{2,n}$ admits Triple Even Star Decomposition $\{3S_2, 3S_4, \ldots, 3S_{4s}\}$ wheres and n are odd & $n = 3s(2s+1)$.

Thus $E(K_{2,n}) = E(K_{2,n-1}) \cup E(K_{1,2})$.

$$E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s-2}) \cup E(S_{4s-2}) \cup E(S_{4s-2}) \cup E(S_2).$$

$$E(K_{2,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \ldots \cup E(S_{4s-2}) \cup E(S_{4s-2}) \cup E(S_{4s-2})$$

Hence $K_{2,n}$ admits Triple Even star Decomposition $\{3S_2, 3S_4, \ldots, 3S_{4s-2}\}$ where s and n are odd & $n = 3s(2s-1)$.

3. Triple Even Star Decomposition of $K_{3,n}$

In this section, we give characterization for $K_{3,n}$ to be Triple Even StarDecomposable.

Theorem 3.1. Let n be any positive integer with $n > 3$. Then $K_{3,n}$ admits Triple Even Star Decomposition $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$ [3k-decomposition]with $k = 3s$ or $k = 3s-1$, $s \in N$ iff $n = 3s(3s \pm 1)$; $s \in N$.

Proof. Let $G = K_{3,n}$ with $n \in N$ and $n > 3$. Then $|E(K_{3,n})| = 3n$. Assume that $K_{3,n}$ has a TESD $\{3S_2, 3S_4, \ldots, 3S_{2k}\}$. Clearly $3n = 3k(k+1)$ where k denotes the total number of decompositions. Thus, $3n = 3k(k+1) \Rightarrow n = k(k+1)$. Suppose $k = 3s$, then $\Rightarrow n = 3s(3s+1)$. Suppose $k = 3s-1$,then

$\Rightarrow n = (3s-1)(3s-1+1) \Rightarrow n = 3s(3s-1)$. Therefore $n = 3s(3s \pm 1)$.

Conversely assume that $n = 3s(3s \pm 1), s \in N$.

Let $K_{m,n} = (V_1(G), V_2(G))$ where $V_1(G) = \{1, 2, \ldots, m\}$ and $V_2(G) = \{1', 2', \ldots, n'\}$.

Consider the matrix (a_{ij}) with order $\begin{cases} 3 \times 3s & \text{if } n = 3s(3s+1) \\ 3 \times (3s-1) & \text{if } n = 3s(3s-1) \end{cases}$ as follows:

Define $a_{1j} = a_{2j} = a_{3j}$ for all j with $a_{11} = \begin{cases} 6s & \text{if } n = 3s(2s+1) \\ 6s-2 & \text{if } n = 3s(2s-1) \end{cases}$ $a_{3(j+1)} = a_{3j} - 2$ where $\begin{cases} 1 \leq j \leq 3s & \text{if } n = 3s(3s+1) \\ 1 \leq j \leq 3s-1 & \text{if } n = 3s(3s-1) \end{cases}$

$a_{1(j+1)} = a_{1j} - 2;\; 2 \leq j \leq 3s.$

Clearly numbers of entries are $\begin{cases} 9s & \text{if } n = 3s(2s+1) \\ 9s-3 & \text{if } n = 3s(2s-1) \end{cases}$ and sum of each row is n. Also entry starts with $\begin{cases} 9s & \text{if } n = 3s(2s+1) \\ 9s-3 & \text{if } n = 3s(2s-1) \end{cases}$ and ends with minimum entry 2. Thus we have different entries $\begin{cases} 2, 4, 6, \dots, 6s & \text{if } n = 3s(3s+1) \\ 0, 2, 4, 6, \dots, 6s-2 & \text{if } n = 3s(3s-1) \end{cases}$ in the matrix (a_{ij}) with each entry repeated thrice with identical rows.

Now edgesincident with i can be decomposed into $3s$ stars

$S_{a_{i1}}, S_{a_{i2}}, \dots, S_{a_{i(3s)}}$ or $(3s-1)$ stars $S_{a_{i1}}, S_{a_{i2}}, \dots, S_{a_{i(3s-1)}}$ for $\begin{cases} 1 \le j \le 3s & \text{if } n = 3s(3s+1) \\ 1 \le j \le 3s-1 & \text{if } n = 3s(3s-1) \end{cases}$ where

$S_{a_{ij}} = (i; [a_{i1} + a_{i2} + \dots + a_{i(j-1)} + 1]', [a_{i1} + a_{i2} + \dots + a_{i(j-1)} + 2]', \dots,$

$[a_{i1} + a_{i2} + \dots + a_{ij}]'); \boldsymbol{j} = 1, 2, \dots, 3s$ (or) $\boldsymbol{j} = 1, 2, \dots, 3s-1$ provided

$n = 3s(3s+1)$ or n =3s(3s-1) respectively.

Thus $E(K_{3,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \dots \cup E(S_{6s}) \cup E(S_{6s}) \cup E(S_{6s})$ when $n = 3s(3s+1)$ if $a_{11} = 6s$

Thus $E(K_{3,n}) = E(S_2) \cup E(S_2) \cup E(S_2) \cup E(S_4) \cup E(S_4) \cup E(S_4) \cup \dots \cup E(S_{6s-2}) \cup E(S_{6s-2}) \cup E(S_{6s-2})$ when $n = 3s(3s-1)$ if $a_{11} = 6s-2$.

Hence $K_{3,n}$ admits Triple Even Star Decomposition $\{3S_2, 23_4, \dots, 3S_{2k}\}$ with $k = 3s$ when $n = 3s(3s+1)$ or $k = (3s-1)$ when $n = 3s(3s-1)$; $s \in N$.

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