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Comprative Study Kirchoff's and Tutte Matrix Theorem

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Abstract: In the mathematical field of graph theory, there are so many topic. In general the Cayley's formula which provides the number of spanning tree in complete graph. There are two theorem Kirchoff's Matrix theorem and Tutte Martrix theorem are also used to count spanning tree of a graph. In this paper we focus to count number of spanning in a graph with help of both theorem. At last we draw conclusion that Tutte Matrix theorem approach to compute the number of spanning arborescence of graph also.

Key Word : *Laplacian Matrix, Digraph, Adjacency Matrix, incidence matrix, Undirected graph.*

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I. Introduction :

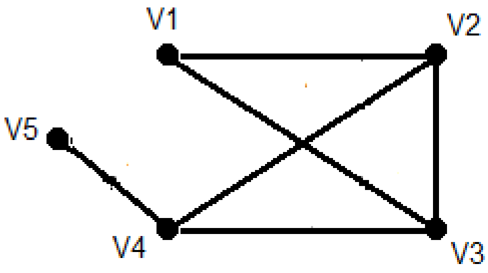
Let us consider a simple graph means without no loop & paralled edges $G = (V, E)$ where V is the set of vertices & E is set of edges each of whose element is a pair of distinct vertices.We can assume that we will familiar with basic concept graph theory. Let $V = \{1,2,3, \dots, n\}$ & $E = \{e_1, e_2, \dots\dots e_n\}$.Then adjacency matric $A(G)$ of G is $n \times n$. Matrix with its row and columns indexed by V with the (i, j) eintry equal to 1 if vertices i, j are adjacent and 0 otherewise.

Thus $A(G)$ is symmetry matrix its i^{th} row or column sum equal to $\deg(v_i)$ which define as degree of vertex. Let $D(G)$ denoted the $n \times n$ diagonal matrix whose i^{th} diagonal entry is $di(G), i = 1, 2, \dots, n$. Then Laplaein matrix of G denoted by

$L(G)$ is define as

$$L(G) = D(G) - A(G)$$

For example Let a graph G.



The vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the edges set $E = \{e_1, e_2, e_3, e_4, e_5\}$ then

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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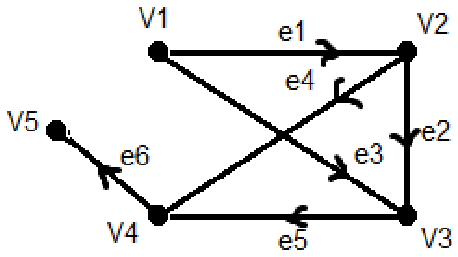
$$D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L(G) = D(G) - A(G) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

There is another way to represent Laplacian matrix. Let now consider G is a digraph.



Then we can take an incidence matrix of G is $Q(G)$ of $n \times m$. The row and column of $Q(G)$ is 0. If vertex i and edges e_j are not incident other wise it is 1 or -1 for example fig.

$$Q(G) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we can determine $Q(G)^T$. If we define $Q(G).\,Q(G)^T$ then it is equal to $L(G)$. So we can say

$$L(G) = Q(G).\,Q(G)^T$$

Then we can describe some basic properties of Laplacian Matrix.

(i) Laplacian Matrix is symmetric matrix

(ii) The non-diagonal element of Laplacian matrix is non-positive, mean non-diagonal element is either 0 or -1. That implies Laplacin matrix is Stieltjes Matrix.

Stieltjes Matrix : A matrix which all non-diagonal element is either 0 or negative and value of that matrix is positive then that matrix is known stieltjes Matrix.

(iii) The rank of $L(G)$ is $(n - k)$, where k is number of connected component of G In particular if G is connected then rank of $L(G)$ is $(n - 1)$.

There are so many properties of Laplacian matrix known but it this papere we focused Kirchhoff's matrix theorem,which useful to determine spanning tree or tree nature.

II. Kirchhoff's Matrix-Tree Theorem :

It is very beautiful theorem that useful to count spanning tree in graph. It describe very good connection between graph theory and linear algebra. The result discovered by German Physicist Gustav Kirchhoffs in 1847 during study of electric circuit. We well known about spanning tree that if a subgraph H of a Graph G contain every vertices of Graph G and that subgraph has no any cycle that such subgraph is known as spanning tree.

We can also define a Laplacian matrix another way

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$$L_{ij} = \begin{pmatrix} \deg(v_j) & if\ i = j \\ -1 & if\ i \neq j\ and\ (v_i v_j) \\ 0 & otherwise \end{pmatrix}$$

It is equivalently $L = D - A$.

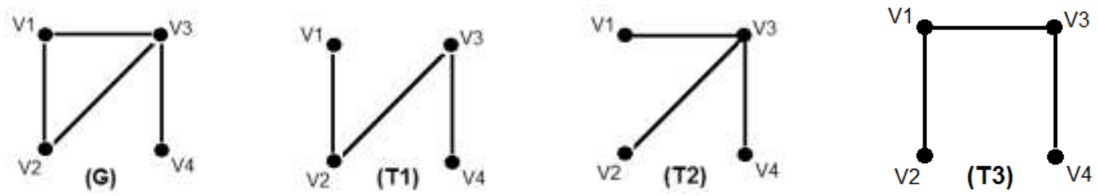
Theorem : It $G(V, E)$ is an undirected graph and L is its Laplacian matrix, then number of spanning tree (N_T) contained in G is determine by following computation.

(i) Chosen a vertex (V_i) and eliminate the i^{th} row and i^{th} column from L to get new matrix L_i.

(ii) Compute $N_T = \det(L_i)$

where N_T count spanning tree that are distinct as subgraph of G. Thus some of the tree that contribute to N_T may be isomorphic.

<u>For example</u>



If G we take G to be the graph whose Laplacian is

$$L = D - A$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

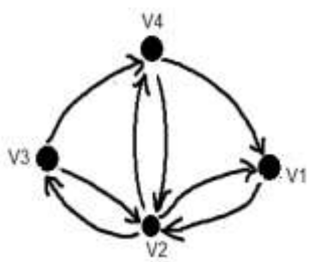
$$= \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Choose $v_j = v_i$, we get

$$L_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow N_T = \det(L_1) = 3$$

The number N_j is count spanning arborescences that are distinct as subgraphs of G equivalently we regard the vertices as distinguishable. Hence some of the arborescences that contribute to N_j may be isomorphic,but if they involve different edges we will count them separately.

For example



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$$L = D_{in} - A$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

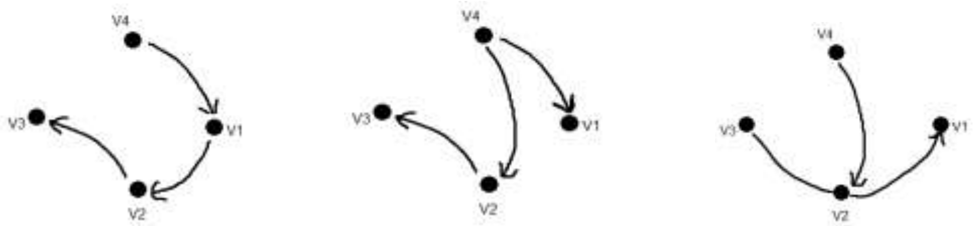
$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \Rightarrow \det(N_i) = 2$$

Similarly we can determine N_2, N_3, N_4 as

$$\det(N_2) = 4, \quad \det(N_3) = 7, \ \det(N_4) = 7.$$

We can draw N_4 (means vertex V_4) spanning tree.



Above three spanning tree along the vertex V_4 like as an arborescences spanning tree.

This is alternative, note that as Cayley's formula count's the number of distinct tabled trees of complete graph K_n. We need to compute any cofactor of the Laplacian matrix K_n. The Laplacian matrix in this case

$$L_1 = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & & & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

Any cofactor of the above matrix is n^{n-2}, which is Cayley formula.

III. Tutte Matrix-Tree Theorem :

After Kirchhoff result in 1948. W. T. Tutte discovered a result for directed graph or diagraph. To study that result we can define some important definition.

Definition : A vertex $v \in V$ in a diagraph $G(V, E)$ is a root if very other vertex is assessable from v.

Definition : A graph $G(V, E)$ is directed tree or arborescence if G contain a root and the graph G that one obtains by ignoring the directedness of the edges is a tree.

Definition : A subgraph $T(V, E^1)$ of a diagraph $G(V, E)$ is a spanning arborescence if T is arborescence that contain all the vertices G.

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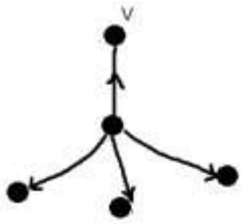

The graph is an arborescence whose v is root vertex.

Theorem : If $G(V,E)$ is a diagraph with vertex set $v = \{v_1, v_2, \ldots, v_n\}$ and L is an $n \times n$ matrix whose entries are given by

$$L_{ij} = \begin{cases} \deg(in)(v_j) & if\ i = j \\ -1 & if\ i \neq j\ and\ (v_i v_j) \in E \\ 0 & otherwise \end{cases}$$

Then number of spanning arborescence with root v_j is

$$N_j = \det(L_j)$$

IV. Conclusion :

Details of above both theorem are long. So we will describe some part of both theorem in this paper. One of these is the connection between Tutte directed matrix tree theorem and Kirchoff's undirected version. The idea is come in above figure when we want to count spanning tree an undirected graph. If we make any undirected graph to directed graph with same vertex set but two running directed edges means one running in each direction for each of edges. Means if G is an undirected graph with edge $e = (a,b)$ then H has a directed graph with edge e is (a,b) and (b,a). We choose an arbitrary vertex v in H and count the spanning arborescences that have v as root. It is not necessary that each spanning tree in G coresponds to a unique v −rooted arborescence in H and vice-versa means there is a bijection between the set of spanning trees in G and v −rooted spanning arborescences in H. How this bijection acts we will discuss in next paper.

Finally, noted that for our directed graph H which includes the edges (a,b) and (b,a). For undirected graph contain (a,b) we have

$$deg_{in}(v) = deg_{out}(v) = deg_G(v)\, v \in V$$

where $deg_{in}(v)$ and $deg_{out}(v)$ are in H and $deg_G(v)$ is G. Means result is that the matrix L appearing in Tutte's Theorem is equal element by element to the graph Laplacian appearing in Kirchoff's theorem, So if we use Tutte approach to compute the number of spanning arborescence in H. The result will be the same numerically as if we have used Kirchoff's theorem to count spanning tree in G.

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