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An Overview of the Different Kinds of Vector Space Partitions

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Abstract:

In a finite vector space $V(n,q)$, where V is n-dimensional over a finite field with q elements, a collection P of subspaces is called a vector space partition. The property of this set P is that any vector that is not zero may be found in exactly one element of P. Partitions of vector spaces have strong ties to design theory, error-correcting algorithms, and finite projective planes.

The first portion of my talk will focus on the mathematical fields that share common ground with vector space partitions. The rest of the lecture will go over some of the most well-known results on vector space partition classification. Heden and Lehmann's result on vector space partitions and maximal partial spreads, as well as El-Zanati et al.'s recent findings on the types found in spaces $V(n, 2)$ for n = 8 or less, the Beutelspacher and Heden theorem on T-partitions, and their newly established condition for the existence of a vector space partition will all be covered. Furthermore, I will demonstrate Heden's theorem about the tail length of a vector space split. Finally, I shall provide some historical notes.

Keywords: Vector spaces, Finite fields, Projective planes, subparts spreads, T-Partitions.

1. Introduction:

We will primarily focus on vector spaces that have a limited number of dimensions and are defined over finite fields. Additionally, we will examine collections of subspaces that span the whole vector space and only cross at the zero vector. When n is the size of V and q is the number of components in the scalar field, a configuration in the space of vectors "V = V(n, q)" is defined as a vector space partition P consisting of subspaces.

$$U, U' \in P \implies U \cap U' = \{\bar{0}\}$$

And $$V = \cup_{U \in P} U.$$

First, let us provide two significant instances of vector space partitions that are not trivial.

For Instance-1:

Consider the equation $q = p^k$, where q is "any power of a prime number p. Let's define the finite field $''F''$ as $GF(q^4)$, which may be seen as a 4-dimensional vector space $''V''$ over the finite field $GF(q)$. Let $\alpha_1, \alpha_2, \alpha_3, \dots \dots . \alpha_K$ be an array of coset representations of the multiplicative group of a subfield $GF(q^2)$ in the multiplicative group of F," where k is defined as $(|F| - 1)/(|GF(q^{2)} - 1|$. The subsequent set of subspaces inside V.

$$P = \{ \alpha_1 GF(q^2), \alpha_2 GF(q^2), \ldots, \alpha_K GF(q^2) \}$$

will form a partition of V which is a vector space. The subsequent establishment of "a vector space" separation is credited to Dekker [9] and distinctly "Beutelspacher" [4].

For Instance-2:

Take into consideration "the finite field $GF(q^k)$ as a vector space W over $GF(q)$ and let U be a subspace of W." We create a subspace $U\alpha$ of the space "$V = W \times U$" by using the formula "$U\alpha = \{(\alpha u, u) \mid u \in U\}$" for every instance of "$\alpha \in GF(q^k)$".

The following set P of subspaces to V

"$P = \{ U\alpha \mid \alpha \in GF(q^k) \} \cup \{ W \times \{\bar{0}\} \}$"

will make up a division of V in vector space. One way to describe a partition P in a vector space is as a subtype

$$[d_1^{n1} d_2^{n2} \ldots \ldots d_t^{nt}],$$

"if P consists of n_1 spaces of dimension d_1, n_2 spaces of dimension d_2, etc., where $d_1, d_2, d_3, \ldots d_t$ are t distinct non-negative integers. So, the partition in Example 1 is of type $[2q^2 + 1]$ and the partition in Example 2 is, in case $dim(W) \neq dim(U), of\ type$ $[\dim(W)^1\, dim(U)^{|W|}]$."

Discussing the many kinds of vector space partitions that are feasible may seem to be a simple undertaking. However, if you use finite vector spaces with a dimension of eight "over a finite field that contains two elements," you will encounter significant difficulties. This specific case is the "first" open case that has been opened. The problem of the various "types of vector space partitions was investigated in the 1970s and 1980s by Bu [9], Beutelspacher [4], and Heden [22]. Additionally, during this millennium, El-Zanati, Seelinger, Sissokho, Spence, and Vanden Eynden made numerous contributions to the study of this problem, as evidenced by the publication [15], as well as by Heden and Lehmann [29]. This article's primary objective is to provide a comprehensive analysis of the most current findings, in addition to the findings from the 1970s and 1980s, concerning the topic at hand. In the first few parts, we will briefly explore the relationship "between vector space partitions and projective planes and the relationship to error-correcting codes." This provides a reason for studying "vector space partitions," which we will investigate. The last section will include historical notes on vector space partition difficulties and group partition problems. These remarks will be included in the preceding section.

2. Projective planes and partitions in vector space

A projective plane is a mathematical structure that is made up of lines and points. It is defined by a set of criteria that these lines and points must meet.

1. The intersection of any two lines occurs at a single, distinct location.
2. Every pair of points lies on exactly one line.
3. There are four points such that each point is only connected to at most two lines.

The counting reasons demonstrate "that the number of points will be equivalent to the number of lines, namely the integer $q^2 + q + 1$. The number q will be referred to as the plane's order. From every given point, there are precisely $q + 1$ lines passing through it, and each line has exactly $q + 1$ points."

By removing a single line, known as the "line at infinity," along with its corresponding points, we may derive an affine plane from any projective plane. The remaining elements will be composed of parallel sets of lines. On the other hand, there is only one line for every pair of remaining points, and each point is included in exactly one member of each concurrent class. However, by including a line at infinity, we may connect any projective plane to any affine plane. In this line, the various dots stand for the various parallel classes.

In this paper, we will describe a technique for building predictive planes utilizing vector space partitioning that was proposed by Dekker [10]. Consider a 4-dimensional vector space, denoted as "$V = V(4, q)$", over a finite field with q elements. Consider a vector space partition, denoted as P, of V that only consists of 2-dimensional subspaces of V:

The set P is defined as $\{U1, U2, ..., Ut\}$, where the dimension of each U_i is 2, for "$i = 1, 2, ..., t = q^2 + 1.$"

(Up until now, we have only explored the basic division of every "2-dimensional space into 1-dimensional spaces.")

An affine plane is initially generated by this division. Four distinct vectors from V will make up the points. In the partition, the lines will represent the cosets of the spaces. In other words, each line Li,α may be expressed as $\alpha + Ui$, where α is an element of $GF(q4)$.

Duplications may arise, and "each element Ui of the vector space partition forms a parallel class including $q4 / q2 = q2$ lines, since separate cosets of subgroups are disjoint and together include the whole space. To demonstrate the existence of a single line passing through any two points and in the affine plane $GF(q4\ )$, it suffices to locate U_i such that $U_i P'$."

The equation "$L_{i,}\beta = \beta + U_i$" includes both the point "$\beta + 0 = \beta$" and the point "$\beta + (\alpha - \beta) = \alpha.$" A similar procedure is used to confirm the other properties of an affine plane. Just as we demonstrated before, by including a line at infinity to this algebraic plane, we can now build a projective plane.

To get projective planes with different properties, we use the Drekkere[1] architecture in conjunction with different vector space partitions. A Desarguesian projective plane may be obtained by following the vector space partition described in Example 1. Nevertheless, obtaining "a Non-Desarguessian projective plane" by the use of similar vector space partitions is a rather simple task.

A partition of the "vector space" "$V\ (4, q)$" into mutually disjoint lines, denoted as a spread or line spread, is referred to as a vector space partition of the kind "$[2q^2+1]$". The spread consists of a family of lines that cover "all points in the projective space $PG(3, q)$ of dimension 3."

It is important to note that not every projective plane can be obtained in this manner. For instance, Dembowski's canonical work [11] provides examples of projective planes that cannot be discovered using this approach.

3. Maximum subpart spreads

Determining the existence of a projective plane of order q that is not a power of a prime is a very complex and significant issue, but it is undeniably intriguing. There is no existence of projective planes with orders 6, 10, and 14, as well as for a subsequent endless series of integers. To be more exact, the Bruck-Ryser-Chowla theorem [6] provides the sole known global limitation on the order of the projective plane. It states that if the order n is equivalent to 1 or 2 modulo 4, it must be the result of adding two perfect squares. For example, the number 14 cannot be expressed as the sum of two squares. The first unresolved instance is when q equals 12.

In order to create such a plane, we may begin by establishing a set of parallel lines that are organized into distinct groups, resulting in what is known as a partial net. Every point on the plane must be part of one line in each set of parallel sets for a net to be

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considered valid. A partial net is formed when all lines in a collection of identical line families are removed from an affine plane.

Bruck [5] proved in 1963 that there is a limit N(n) such that an algebraic plane of order n may be formed by augmenting a partial net with more concurrent classes than N(n).. To be more specific, consider the function

$$"p(x) = \frac{1}{2}x^4 + x^3 + x^2 + \frac{3}{2}x,"$$

and let "$d = n - 1 - t$," "where t is the number of parallel classes in a partial net." If the inequality "$p(d - 1) < n$" is true, it is possible to extend the partial net to form an affine plane. In simpler terms, if you are able to discover "9 mutually orthogonal latin squares of order 12, then you may add another 2 to form a set of 11 mutually orthogonal latin squares of order 12. This collection is large enough to represent a projective plane of order 12."

The logical extension of these cases to the field of vector space partition problems is to study how parallel classes of lines formed from two-dimensional subspaces inside a four-dimensional vector space emerge.

A maximum "partial spread is a set S of 2-dimensional subspaces of $V = V(4, q)$ such that every 2-dimensional subspace of V intersects with at least one member of S in a non-trivial way."

The first person to look at maximal partial spreads was Mesner [37]. He gave his students apparently random instructions in 1967 to choose two-dimensional zones with non-overlapping areas. His kids could always find a method to expand the lines they had previously identified to make a whole layout if they found more than a certain limit. "Maximal partial spreads have been extensively investigated by many writers, including Mesner [37], Bruen [7], Bruen and Thas [8], Heden [24], Blokhuis [6], Heden, Pambianco, Marcugino, and Faina [28], Blokhuis and Metsch [3], Ebert [13], Beutelspacher [4], G´acs and Sz¨onyi [18]. The most well recognized upper limit for a maximum partial spread is established by Blokhuis [3]": For any maximum "partial spread S in $V\ (4, p)$, where p is a prime" integer,

$$"|S| \leq p^2 - \frac{p+1}{2}."$$

"Bruen and Thas, Beutelspacher Ebert, and several other scholars have created maximal partial spreads" with sizes "$q^2 - q + 1$" and "$q^2 - q + 2$" Bruen and Thas [8] have conjectured that "$q^2 - q + 2$" is the maximum size for a non-trivial maximal partial spread in $V(4, q)$. Nevertheless, Heden [25] demonstrated in 2000 that this claim is incorrect. He used "a computer search to discover a greatest partial spread of size 45 in $V\ (4, 7)$. Now, we examine the architecture in Example 2 under a very specific but significant scenario."

For Instance-3:

The direct product $W \times U$ refers to the combination of two sets, where W represents "the finite field *GF(32)"* and is considered "a vector space over *GF(2)."*The subspace U will have a size of 3 in the vector space W. "By using the construction of Bu, as seen in Example 2, we get a vector space partition of the form $[3^{32}5^1] of\ V(8\ ,2)$" Next, we proceed to divide the subspace W, which has a dimension of 5, into a subspace with a size of 3 and the other subspaces with a dimension of 1 each. Thus, we have a partition of the form $[1^{24}3^{33}]$. The spread will consist of three elements and have a size of 33 in the $V\ (8, 2)$ space.

If the hypothesis proposed by "Eisfeld and Storme, as well as by Hong and Patel" [31], is shown to be accurate, then it would imply that the partial 3-spread in $V\ (8, 2)$ is of the maximum conceivable magnitude. "Recently, El-Zanati et al [15]

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discovered a vector space partition of the form$[1^{17}3^{34}]$" by a computer search. The significance of this vector space split will be relevant in the subsequent discussion.

The aforementioned example substantiates the arduousness of identifying all possible vector space divisions. It is worth noting that there has been significant research on maximum partial t-spreads in $V\ (n, q)$. The research has received contributions from Beutelspacher [4] in 1980 and from Govaerts and Storme [19] throughout this millennium.

4. Flawless codes and vector space partitions

A direct product of sets contains a subset C that is an ideal e-error correcting code.

$$"C \subseteq A_1 \times A_2 \times \ldots \times A_t"$$

where each "word x in this direct product varies in at most e coordinate points from a unique word in C. The study of error correcting codes," including both perfect and non-perfect codes, is a well-established field that emerged in the late 1940s amid the advancement of computers. However, it has significant relevance, particularly in the context of information transmission.

In 1972, Herzog and Schönheim [30] discovered "that vector space partitions might be used to create flawless 1-error correcting codes. Consider a vector space partition" denoted as

$$"P = \{ U_1, U_2, \ldots, U_t \}"$$

of the vector space "$V = V\ (n, q)$." The map φ, defined as

$$\text{“}\varphi: U_1 \times U_2 \times \ldots \times U_t \rightarrow V,\text{”}$$

where

$$(u_1, u_2, \ldots, u_t) \rightarrow u_1 + u_2 + \ldots + u_t .$$

The kernel of this map, i.e.

$$\ker(\varphi) = \{(u_1, u_2, u_3, \ldots\ldots, u_t) | u_1 + u_2 + \cdots. + u_t| = 0\}$$

is "a perfect 1-error correcting code." If the spaces in the vector space partition P have different sizes or dimensions, the codes are referred to as "mixed perfect codes."

For those who are well-versed in the traditional Hamming code, the above structure could seem like an expansion of the famous Hamming code as the H-matrix kernel. For instance, consider

$$H = \begin{pmatrix} 0\ 0\ 0\ 1\ 1\ 1\ 1 \\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \end{pmatrix}$$

From the vector space division that includes the subspaces of V (3, 2), the space containing the null of this matrix, the Hamming code, may be obtained in the following way:

$$P = \{ U1 = \{(0,0,0),(0,0,1)\}, U2 = \{(0,0,0),(0,1,0)\}, \ldots, U7 = \{(0,0,0),(1,1,1)\} \}$$

"All perfect codes that are built in this manner will, in fact, be linear codes, which are vector spaces over the field GF(q)." Last but not least, Herzog and Sch¨onheim [30] made the observation that any "linear perfect 1-error correcting code comes from a vector space partition," as was explained before.

5. Addressing the classifications of vector space partitions

As previously stated, and maybe inferred "from Example 3 above, determining all potential forms of vector space partitions is a challenging endeavour." It encompasses both the discovery of novel structures and the establishment of essential prerequisites for the existence of a certain category. Thus yet, no comprehensive set of circumstances that are both required and sufficient has been discovered.

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5.1.Essential prerequisites:

In this section, we will consistently assume that if a vector space partition is of the form $d_1^{n_1} d_2^{n_2} \ldots\ldots . . d_k^{n_k}$ then the values of d1, d2, ..., dk are in ascending order.

In order for "a vector space partition of type" $d_1^{n_1} d_2^{n_2} \ldots\ldots . . d_k^{n_k}$ to exist in "$V = V(n, q)$", the following packing condition must be true: every vector is included in only one space of the partition.

$$n_1(q^{d_1} - 1) + n2(q^{d_2} - 1) + \ldots + n_k(q^{d_k} - 1) = q^n - 1$$

"For any two members *U and W* of a vector space partition" "$P\ of\ V = V(n, q)$", it is established that the dimension of the span of the union of *U and W* is equal to the sum of the dimensions of *U and W*. This observation was initially made by Bu [9]. Therefore, for any *i and j:*

$$\text{“}d_i + d_j \leq n.\text{”}$$

This criterion shall hereafter be referred to as the dimension condition. Example 2 demonstrates the possibility of achieving equality.

Kurz [35] noted that the packing criterion implies that,

$$\frac{q^n - 1}{q^{d_k} - 1} \leq \sum_{i=1}^{k} n_i = |P| \leq \frac{q^n - 1}{q^{d_k} - 1}$$

A constraint "for the number of spaces in a vector space partitioning was improved by Heden and Lehmann [29]. This bound states that if we are not in the situation of Example 2 of Section 1, then"

$$"|P| \geq q^{d_k} + q^{d_{k-1}} + 1."$$

We relied on the idea of the second packing condition, which indicates that a vector space partition PH is obtained by intersecting all the spaces of a vector space partition P with a hyperplane H. It will be more convenient to use the following notation to represent the second packing condition:

A "$(m_k, m_{k-1}, \ldots, m_2, m_1)$" −partition is equivalent to a "$[1^{m_1}\ 2^{m_2} \ldots k^{m_k}]$" −partition, with the allowance of zero values for "some of the non-negative integer exponents."

"A hyperplane *H* is classified as type *b = (bk, . . . , b2, b1)* if it includes bi of the subspaces of dimension *i of P*. Let s_b be the quantity of hyperplanes classified as type *b*."

Heden and Lehmann [26] developed the second packing condition, which is

$$s_b \neq 0 \implies \sum_{d=1}^{k} b_d q_d = \sum_{d=1}^{k} m_d - 1.$$

"Let B be the set of all possible solutions that satisfy the diophantine equation mentioned above." Heden and Lehmann [29] demonstrated the following required requirements by considering incidents in a way that accounts for them twice:

For all $1 \leq d, d' \leq n - 2,$

$$\sum_{b \in B} b_d S_b = m_d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\sum_{b \in B} \binom{b_d}{2} s_b = \binom{m_d}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\sum_{b \in B} b_d b_{d'} s_b = m_d m_{d'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Heden and Lehmann [29] developed the following by using the hyperplane criteria, which are essential requirements. Denote the function *"V"* as *"V(2t, q)"* and suppose that *"V"* is divided into parts of the form *"(mt, ..., m1)"*, where *"mt"* is equal to *"q^t + 1 - a."* Let *"d is less than t,"* such that *"md is greater than 0."*

$$m_d < \frac{q^t - 1}{q^{t-d} - 1} \ldots\ldots\ldots\ldots (4)$$

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Hence,

$$"a \geq m_d - R_q(t,d,m_d) \quad \dots\dots\dots\dots\dots\dots\dots (5)"$$

Whence,

$$R_q(t,d,m) = m(m - 1)\frac{\frac{1}{2}(q^{2t-2d} - 1) + 1 - q^{t-d}}{q^t - 1 - m\,(q^{t-d} - 1)}$$

Now, we will assess this constraint in a specific scenario. We regard the expression as "$V\ (2t, q)$." In Example 1 of Section 1, we can readily identify a spread that comprises "$q^t + 1$" spaces with a size of t. Subsequently, we may replace a certain quantity "of these spaces by partitions of vector spaces," which are composed of spaces of smaller dimensions. If the condition "$t/2 < d < t$" is satisfied, it is not possible to "get a vector space partition with more than a subspace of size d. Heden and Lehmann [29]" investigated the possibility of obtaining more than d-dimensional spaces, while still having $q^t + 1 - a$ spaces of dimension t. However, these t-dimensional spaces would be different from the ones in a full spread. Nevertheless, they proved that this situation is impossible to achieve if there is a finite number of spaces of dimension d. More specifically:

Assume that d is equal to the product of t *and* k, and that m_d is the number of spaces of d dimensions. If

$$m_d \leq \sqrt{2q^{(t-2k)/2}}$$

Hence,

$$"a \geq m_d"$$

To conclude this part, we will now address the tail's length. The set of spaces with the shortest conceivable dimension is called the tail of a vector space partition. You may measure the tail's magnitude by measuring its length. As we'll see in a bit, Example 3's vector space division has a tail magnitude of 17. Following Section 4's description, Heden [27] used the relationship with perfect codes to determine the following limits on the length of a vector space partition's tail. For any vector space partition of the form $d_1^{n_1}, d_2^{n_2}, \dots. d_k^{n_k}$,

(i) If the expression "q^{d2-d1} does not divide n_1 and if $d_2 < 2d_1$," then the expression "$n_1 \geq q^{d1} + 1$" is true.

(ii) If the expression "q^{d2-d1} does not divide n_1 and $d_2 \geq 2d_1$, then either d_1 divides d_2 and $n_1 = (q^{d_2 - 1})/(q^{d_1 - 1})$ or $n_1 > 2q^{d_2 - d_1}$" holds true.

(iii) If the quotient of dividing n_1 by "$q^{d_2-d_1}$" is a whole number and "d_2" is less than twice "d_1", then "n_1" is greater than or equal to "$q^{d_2} - q^{d_1} + q^{d_2-d_1}$".

(iv) iv) If the expression "$q^{d_2-d_1}$" divides n1 and "$d_2 \geq 2d_1$," then the statement "$n_1 \geq q\ d_2$" is true.

5.2.T-partitions

A vector space partition refers to the division of a vector space into non-overlapping subsets, where each subset contains vectors that have certain properties or characteristics. A set P is considered a T-partition if

$$"T = \{\, dim(W) \mid W \in P\}"$$

The objective is to determine the necessary conditions for "a given set T of positive integers to ensure the existence of a T-partition." It is usually assumed that the set "T" consists of elements "$t_1, t_2, \dots, t_k$" wherein "t_1" is less than "t_2" and so on.

For the scenario when "$t_1 = 1$, it is straightforward to identify a T-partition for any space $V = V(n, q)$ where $t_k + t_{k-1} \leq n$. When we treat V as a direct product of W *and* U, where $d_2 = dim(W) = t_k$ *and* $d_1 = dim(U) = n - dim(W)$, we get a partition of

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type $d_1^{q^{d_2}} d_2^1$ by using the framework of Example 2 of a vector space partition from the previous sentence." Now, we divide each of the *"k - 1"* subspaces with dimension d_1 into one subspace with "dimension t_i and the remaining subspaces with dimension 1. It is observed that the inequality $k \le d_2$ *implies that* $k < q^{d_2}$, so this particular T-partition will occur."

For t_1 to be higher than or equal to 2, the situation becomes non-trivial. The idea of T-partition was first out by Beutelspacher [4] in 1978. He made a discovery related to the Frobenius number about the existence of T-partitions.

Consider a set "A" consisting of positive integers, denoted as $\{a_1, a_2, \ldots, a_k\}$. "Assume that the greatest common divisor of these numbers is 1. The Frobenius number $g(A)$ is defined as the largest integer n that cannot be expressed as a linear combination" of the numbers in set A, using non-negative coefficients. Kontorovich [34] demonstrated that the inequality

$$g(A) \le 2a_1 \left\lfloor \frac{a_k}{k} \right\rfloor - a_1 \text{ holds,}$$

where "a_1" represents the lowest integer and a_k" represents the biggest integer in set A.

Based on the aforementioned result from Selmer, Beutelspacher [4] demonstrated the following. Let's examine the vector space denoted as "$V = V(n, q)$." For

$$n > 2t_1 \left\lfloor \frac{t_k}{d.t_k} \right\rfloor + t_2 + \cdots + t_k ,$$

A set "V has a T-partition if and only if the greatest common divisor of the elements in T" divides the number n.

It is important to note that if $V(n, q)$ can be divided into T parts, then the greatest common divisor of T must divide n.

In addition, Beutelspacher [4] demonstrated the following theorem for T-partitions, specifically when t_1 is equal to 2.

Subsequently, Heden [22] established the conclusion in its entirety. The space V (2t, q) is capable of accommodating a partition with the formula "$T = \{t_1 < t_2 < \ldots < t_k = t\}$.

$T = \{t_1 < t_2 < \ldots < t_k = t\}$."

5.3. An enumeration of the several vector space partitions in V (n, 2), where n is less than or equal to 7.

The "enumeration was conducted by El-Zanati et al in [14]," although Heden [23] had previously eliminated the study of almost all instances. Through a" computer search focused on a specific vector space partition, it was shown that the packing requirement, dimension condition, and tail condition are both necessary and sufficient in the situation of "$V(n \le 7, 2)$." For example, when n is less than or equal to 5," we get the preceding enumeration:

n	*Different types of vector space partitions in* $V(n, 2)$				
1	$[1^1]$,				
2	$[1^3]$,				
	$[2^1]$,				
3	$[1^7]$,	$[1^4 2^1]$,			
	$[3^1]$,				
4	$[1^{15}]$,	$[1^{12} 2^1]$,	$[1^9 2^2]$,	$[1^6 2^3]$,	$[1^3 2^4]$,
	$[2^5]$,				
	$[1^8 3^1]$,				

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	$[4^1]$,							
5	$[1^{31}]$,	$[1^{28}2^1]$,	$[1^{22}2^3]$,	$[1^{19}2^4]$,	$[1^{16}2^5]$,	$[1^{13}2^6]$,	$[1^{10}2^7]$,	$[1^7 2^9]$,
	$[1^{24}3^1]$	$1^{21}2^1 3^1]$,	$[1^{18}2^2 3^1]$,	$[1^{15}2^3 3^1]$,	$[1^{12}2^4 3^1]$,	$[1^9 2^5 3^1]$,	$[1^6 2^6 3^1]$,	$[1^3 2^7 3^1]$,
	$[1^{16}4^1]$,							
	$[5^1]$.							

In their work, Heden, El-Zanati, and colleagues [16] examined the case when n = 8, q = 2, and spaces with dimensions of at least 2 constituted the vector space partitions. Except for one case, all of the packing conditions, size requirements, and tail conditions were determined to be necessary and sufficient in this situation. With these three conditions met, the existence of a vector space partition of type $[2^6 3^6 4^{13}]$ was the only remaining option. Nevertheless, the hyperplane criteria, which were developed for this very situation, might rule out this possibility. It is also possible to infer from Heden and Lehmann's theorem [29] that there is no such vector space split.

5.4. Do any circumstances exist that are both essential and adequate?

By presenting instances, Heden and Lehmann [29] shown that the "packing, dimension, tail, and hyperplane" conditions—the four prerequisites for a vector space partition—are not enough to ensure its existence. In this example, we provide the extraordinary and uncommon instances where we know all the types, according to the size and dimensions of the scalar field.

It can be shown using basic reasoning that $"q^d - 1"$ is a divisor of "q^n - 1" when q is a prime power, if and only if d is a divisor of n. Therefore, according to the initial packing requirement, “a vector space partition of type [d m] can only exist in $V(n, q)$ if d is a divisor of n.” This is likewise satisfactory, since it ensures that $"GF(q^n)"$ has a subfield $"GF(q^d)"$. By using the same method as shown in Example 1, we can easily establish a partition of the vector space of type $"[d^m]"$, where $"m = (q^n - 1)/(q^d - 1)."$

Problem complexity increases, however, even when limiting consideration to two independent degrees in the vector space division, as we go beyond the previously stated simple case. As seen in instance-3, the scenario "$[1^{n_1} d^{n_2}]$" remains little explored. Heden [26] demonstrated the following in his thesis: The conditions of packing, dimension, and

$$"dim(Ui) \geq c, for\ i \leq q,"$$

are both “necessary and sufficient for the existence of a vector space partition” "$U1, U2, \ldots, Uk$" of "$V(n, q)$," given that $"dim(U_{q+1}) = dim(U_{q+2}) = \ldots = dim(U_k) = c."$ An extension of Lindström's theorem [31], this theorem establishes that all subspaces with the exception of one have dimension "c".

6. A few historical observations

George Abram Miller [38] was the first researcher to study these kind of difficulties. In an academic paper published in 1906, he proved that if subgroups can be formed from an abelian group G, then every element of G must have a certain numerical value, the order, which is a "prime number p." The concept behind the proof is straightforward. Let's consider two elements, h_i and h_j, in a group G. Assume that hi has an order of p and h_j has an order of q, where p is a prime number. Additionally, h_i and h_j belong to separate “subgroups, H_i and H_j,” respectively, in “the partition of G.” Furthermore, the equation $"hi + hj = hk"$ belongs to a distinct group called H_k

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inside the partition. The total of h_k, repeated p times, may be expressed as

$$"p \cdot h_k = (h_i + h_j) + (h_i + h_j) + \ldots + (h_i + h_j) = p \cdot h_j ."$$

This total is zero since it is part of both H_k and H_j. Miller found a way to partition a set containing p2 items into subsets with p elements each.

Twenty years after finishing his master's thesis with Miller as his advisor, Rieffel [37] investigated the subgrouping of infinite groups. A Russian researcher named Kontorovich studied and published results on G-partitions with the "unique property: G = HK = KH for any two members H and K in the partition" in 1939 and 1940.

Here is a simple example of a partition within a non-abelian group. Here is one way to divide up the set S3:

$$“S3 = \{id., (1\ 2)\} \cup \{id., (1\ 3)\} \cup \{id., (2\ 3)\} \cup \{id., (1\ 2\ 3), (1\ 3\ 2)\}.”$$

The primary goal of research by Reinhold Baer, his pupil Otto Kegel [33], and Michio Suzuki was to classify subgroups of non-abelian groups.

Takahasi [40] provided a comprehensive overview of this quest for categorization in 2003. If one of the following requirements is satisfied by a group G, then and only then does G have a non-trivial partition in Zappa's view:

1. “G is a p-group when the subgroup HP (G) is not equal to G and the order of G is greater than p.

2. G is a Frobenius group.

3. G is a group of Hughes-Thompson type.

4. G is isomorphic to $PGL(2,p^h)$, where p is an odd prime.

5. G is isomorphic to $PSL(2,p^h)$, where p is a prime.

6. G is isomorphic to a Suzuki group $G(q)$, where $q = 2^h$ and h is greater than 1.”

The Hughes subgroup, formed by G's components without a p-order, is denoted as Hp(G). This is an essential notation to keep in mind. With at least one non-trivial element fixing a point and every non-trivial element fixing precisely one point, a permutation group known as a Frobenius group functions transitively on a finite set. S3 exemplifies a Frobenius group.

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