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Cap Like and Star like Complex Functions

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Abstract: Main aim of this article is the discussion of Univalent complex functions, Cap like complex functions, and star like complex functions.

Keywords: *Univalent function; BEIRBARBACH Conjecture; Cap like function; Star like function; KOEBE's **function;** HADAMARD Product (Convolution).*

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Introduction: We know that a complex valued function is said to be regular or analytic in a domain **D** (a non-empty open connected subset of the complex plane $\mathbb{C}$) if it has a uniquely determined derivative at each point of **D.**

Definition 1: A function $\mathbf{f}(z)$ is said to be a univalent in a domain **D** if $\mathbf{f}(z_1) \neq \mathbf{f}(z_2)$ for all $\{z_1, z_2\} \subset \mathbf{D}$ with $z_1 \neq z_2$.

A necessary condition for analytic function $\mathbf{f}(z)$ to be univalent in **D** is $\mathbf{f}'(z) \neq 0$ in **D.** This condition is not sufficient since $\mathbf{f}(z) = e^z$ is clearly not univalent since $\mathbf{f}(0) = e^0 = 1 = e^{\iota 2\pi} = \mathbf{f}(\iota 2\pi)$ but $\mathbf{f}'(z) = e^z \neq 0$.

By Riemann mapping theorem, one function may map any simply connected domain onto the open unit disc in a one-one conformal manner. Hence, without loss of generality, we confine our attention to the functions that are univalent and analytic in the open unit disc $\{z/ |z| < 1\}$.

Notation: We denote by $\mathbf{U}$ the class of functions $\mathbf{f}(z)$ that are analytic in the open unit disc $\{z/ |z| < 1\}$ and are univalent in an open disc $\{z/ |z| < r \leq 1\}$ with the conditions $\mathbf{f}(0) = 0, \mathbf{f}'(0) = 1$.

BEIRBARBACH Conjecture: In 1916, BEIRBERBACH proved that $|a_2| \leq 2$ for every $\mathbf{f}(z)$ in $\mathbf{U}$ whose Taylor's expansion about the origin is $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ .He also showed that $|a_2| = 2$ for the function $\mathbf{f}(z) = z(1-\kappa z)^{-2}$, $|\kappa| = 1$, which is known as KOEBE's function. Note that singularity of KOEBE's function is $z = \kappa^{-1}$ which is outside the open unit disc $\{z/ |z| < 1\}$ since $|z| = |\kappa^{-1}| = |\kappa|^{-1} = 1$; thus KOEBE's function is analytic in the open unit disc $\{z/ |z| < 1\}$. And $\mathbf{f}'(z) = z(-2)(1-\kappa z)^{-3}(-\kappa) + 1(-2)(1-\kappa z)^{-2}$ implies $\mathbf{f}'(0) = 0(-2)(1-\kappa 0)^{-3}(-\kappa) + 1(1-\kappa 0)^{-2} = 1$.

Clearly $\mathbf{f}(0) = 0$. So KOEBE's function is in $\mathbf{U}$.

Motivated by the extremal property of the KOEBE's function, BEIRBERBACH conjectured that $|a_n| \leq n \quad (n = 2, 3, 4, \cdots)$ for every $\mathbf{f}(z)$ in $\mathbf{U}$. This is known as BEIRBERBACH conjecture which is a challenging problem in mathematics that took almost 70 years to prove it. In 1985, LOUIS BRANZES has proved the conjecture in full.

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Definition 2 : n-th partial sum of the function $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is $s_n(z,\mathbf{f}) = z + \sum_{k=2}^{n} a_k z^k$.

Theorem 1 : Let $\mathbf{f}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$ with $\mathbf{f}(0)=0$, $\mathbf{f}'(0)=1$, and $\mathbf{f}(z)$ is univalent in the disc $\{z/|z|<1\}$, then $s_n(z,\mathbf{f})$ is analytic in disc $\{z/|z|<1\}$ with $s_n(0,\mathbf{f})=0$, $s_n'(0,\mathbf{f})=1$, and is univalent function in $|z|<1-3^{-1}\sqrt{6}<1$ forall integers $n=2,3,4\cdots$.

Proof : Let $\mathbf{f}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$ with $\mathbf{f}(0)=0$, $\mathbf{f}'(0)=1$

$\Rightarrow \quad \mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \Rightarrow \quad s_n(z,\mathbf{f}) = z + \sum_{k=2}^{n} a_k z^k$.

$\Rightarrow \quad s_n(z,\mathbf{f})$ is analytic in the open unit disc $\{z/|z|<1\}$ with $s_n(0,\mathbf{f})=0$, $s_n'(0,\mathbf{f})=1$,

Let $z_1 \neq z_2$. Then $z_1^k \neq z_2^k$. Then $a_k z_1^k \neq a_k z_2^k$ $(k=2,3,4,\cdots)$. But the inequality $z_1 + \sum_{k=2}^{n} a_k z_1^k \neq z_2 + \sum_{k=2}^{n} a_k z_2^k$ may or may not hold. So we can do some work.

Since $z_1 \neq z_2$, we have $z_1 - z_2 \neq 0 \quad \Rightarrow \quad |z_1 - z_2| \neq 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| \le |z_1 - z_2|$ by triangle inequality.

Consider

$$s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f}) = \left[z_1 + \sum_{k=2}^{n} a_k z_1^k\right] - \left[z_2 + \sum_{k=2}^{n} a_k z_2^k\right]$$

$$= z_1 - z_2 + \left[\sum_{k=2}^{n} a_k z_1^k - \sum_{k=2}^{n} a_k z_2^k\right] = z_1 - z_2 + \sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right|$

$$\Rightarrow \quad \left|s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f})\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \ge r - \rho - \left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right|$$

Again by triangle inequality, and by BEIRBERBACH conjecture $|a_k| \le k$ since $\mathbf{f}(z) \in \mathbf{U}$,

$$\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{n} \left|a_k\left[z_1^k - z_2^k\right]\right| = \sum_{k=2}^{n} |a_k|\,|z_1^k - z_2^k|$$

$$\le \sum_{k=2}^{n} k\left[|z_1^k| + |z_2^k|\right]$$

$$= \sum_{k=2}^{n} k\left[|z_1|^k + |z_2|^k\right] \le \sum_{k=2}^{n} k\left[r^k + r^k\right]$$

since $|z_1| \le |z_2| = r$.

i.e. $$\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{n} k 2 r^k = 2r \sum_{k=2}^{n} k\, r^{k-1} = 2r \sum_{k=2}^{n} \frac{d}{dr} r^k = 2r \frac{d}{dr} \sum_{k=2}^{n} r^k$$

i.e. $$\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \le 2r \frac{d}{dr}\left[-1 - r + \sum_{k=0}^{n} r^k\right] = 2r \frac{d}{dr}\left[-1 - r + \frac{1 - r^{n+1}}{1 - r}\right]$$

i.e. $$\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \le 2r\left[0 - 1 + \frac{(1-r)[0-(n+1)r^n] - (1-r^{n+1})(-1)}{(1-r)^2}\right]$$

i.e. $$\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \le 2r\left[-1 + \frac{-(n+1)(1-r)r^n + 1 - r^{n+1}}{(1-r)^2}\right]$$

$$\Rightarrow \quad -\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \ge -2r\left[-1 + \frac{-(n+1)(1-r)r^n + 1 - r^{n+1}}{(1-r)^2}\right]$$

i.e. $$-\left|\sum_{k=2}^{n} a_k\left[z_1^k - z_2^k\right]\right| \ge 2r\left[1 + \frac{(n+1)(1-r)r^n - 1 + r^{n+1}}{(1-r)^2}\right]$$

$$i.e. \quad -\left|\sum_{k=2}^{n} a_k[z_1^k - z_2^k]\right| \geq 2r\left[1 + \frac{(n+1)(1-r)r^n + r^{n+1}}{(1-r)^2} - \frac{1}{(1-r)^2}\right] \geq 2r\left[1 + 0 - \frac{1}{(1-r)^2}\right]$$

since $\quad 0 < r < 1 \quad \Rightarrow \quad -r > -1 \quad \Rightarrow \quad 1 - r > 1 - 1 = 0.$

Thus we have

$$|s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f})| \geq r - \rho - \left|\sum_{k=2}^{n} a_k[z_1^k - z_2^k]\right|$$

$$\geq r - \rho + 2r\left[1 - \frac{1}{(1-r)^2}\right] = 3r - \rho - \frac{2r}{(1-r)^2} = r\left[3 - \frac{2}{(1-r)^2}\right] - \rho.$$

Observe that

$$r\left[3 - \frac{2}{(1-r)^2}\right] - \rho > 0 \quad \Leftrightarrow \quad r\left[3 - \frac{2}{(1-r)^2}\right] > \rho \geq 0$$

Consider

$$0 \leq \rho < r\left[3 - \frac{2}{(1-r)^2}\right] \quad \Rightarrow \quad 3 - \frac{2}{(1-r)^2} > 0 \quad \Rightarrow \quad 3 > \frac{2}{(1-r)^2}$$

$$\Rightarrow \quad (1-r)^2 > \frac{2}{3} \quad \Rightarrow \quad 1 - r > \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \quad \Rightarrow \quad 1 - \frac{\sqrt{6}}{3} > r \quad \Rightarrow \quad r < 1 - \frac{\sqrt{6}}{3} = c$$

$$i.e. \quad 0 \leq \rho = |z_1| \leq |z_2| = r < c = 1 - \frac{\sqrt{6}}{3} = \frac{3 - \sqrt{6}}{3} < 1$$

$$\Rightarrow \quad |s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f})| > r\left[3 - \frac{2}{(1-r)^2}\right] - \rho > 0$$

$$\Rightarrow \quad |s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f})| \neq 0 \quad \Rightarrow \quad s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f}) \neq 0.$$

Hence $s_n(z,\mathbf{f})$ is univalent function in the disc $|z| < c$ for all $n.//$

Definition 3 : A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z| < 1\}$ with $\mathbf{f}(0) = 0,$ $\mathbf{f}'(0) = 1$ is said to be *cap like function* if

$$\mathbf{Re}\left[1 + \frac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] > 0, \quad |z| < 1.$$

Definition 4 : A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z| < 1\}$ and univalent in the open disc $\{z/|z| < c \leq 1\}$ with $\mathbf{f}(0) = 0, \mathbf{f}'(0) = 1$ is said to be *star like function* in the open disc $\{z/|z| < c \leq 1\}$ if

$$\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right] > 0, \quad |z| < c.$$

Theorem 2 : If $\mathbf{f}(z) \in \mathbf{U},$ then $|a_k| \leq 1 \ (k = 2,3,4,\cdots)$ where a_k is coefficient of z^k in taylor series of $\mathbf{f}(z)$; also if $\mathbf{f}(z)$ is cap like function, then $z\mathbf{f}'(z)$ is star like in $\{z/|z| < c \leq 1\},$

$$c = 1 - \frac{\sqrt{6}}{3} = \frac{3 - \sqrt{6}}{3}.$$

Proof : Let $\mathbf{f}(z) \in \mathbf{U} \quad \Rightarrow \quad |a_k| \le k \quad (k = 2, 3, 4, \cdots)$

Put $\mathbf{g}(z) = z\,\mathbf{f}'(z)$ where $\mathbf{f}(z)$ is analytic in unit open disc $\{z / |z| < 1\}$ with $\mathbf{f}(0) = 0$, $\mathbf{f}'(0) = 1$.

Taylor series of $\mathbf{f}(z)$ about the origin is $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} a_k z^k$

$\Rightarrow \quad z\,\mathbf{f}'(z) = z\left[1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right] = z + \sum_{k=2}^{\infty} k a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k$ where $b_k = k a_k$

Let $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0 \quad \Rightarrow \quad |z_1 - z_2| \ne 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| \le |z_1 - z_2|$ by triangle inequality.

Consider

$$\begin{aligned}
\mathbf{g}(z_1) - \mathbf{g}(z_2) &= z_1\,\mathbf{f}'(z_1) - z_2\,\mathbf{f}'(z_2) \\
&= \left[z_1 + \sum_{k=2}^{\infty} k a_k z_1^k\right] - \left[z_2 + \sum_{k=2}^{\infty} k a_k z_2^k\right] \\
&= z_1 - z_2 + \left[\sum_{k=2}^{\infty} k a_k z_1^k - \sum_{k=2}^{\infty} k a_k z_2^k\right] = z_1 - z_2 + \sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]
\end{aligned}$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right|$

$$\Rightarrow \quad |\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \ge r - \rho - \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right|$$

Again by triangle inequality, and by BEIRBERBACH conjecture since

$$\begin{aligned}
\left|\sum_{k=2}^{n} k a_k \left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{n} \left|k a_k \left[z_1^k - z_2^k\right]\right| &= \sum_{k=2}^{n} k |a_k| \, |z_1^k - z_2^k| \\
&\le \sum_{k=2}^{n} kk \left[|z_1^k| + |z_2^k|\right] \\
&= \sum_{k=2}^{n} k^2 \left[|z_1|^k + |z_2|^k\right] \le \sum_{k=2}^{n} k^2 \left[r^k + r^k\right].
\end{aligned}$$

Thus

$$\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2\sum_{k=2}^{\infty} k^2 r^k = 2\sum_{k=2}^{\infty} [k(k-1) + k] r^k = 2\sum_{k=2}^{\infty} k(k-1) r^k + 2\sum_{k=2}^{\infty} k\, r^k$$

i.e. $\quad \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2r^2 \sum_{k=2}^{\infty} k(k-1) r^{k-2} + 2r \sum_{k=2}^{\infty} k\, r^{k-1}$

i.e. $\quad \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2r^2 \sum_{k=2}^{\infty} \frac{d^2}{dr^2} r^k + 2r \sum_{k=2}^{\infty} \frac{d}{dr} r^k = 2r^2 \frac{d^2}{dr^2} \sum_{k=2}^{\infty} r^k + 2r \frac{d}{dr} \sum_{k=2}^{\infty} r^k$

i.e. $\quad \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2r^2 \frac{d^2}{dr^2}\left[-1 - r + \sum_{k=0}^{\infty} r^k\right] + 2r \frac{d}{dr}\left[-1 - r + \sum_{k=0}^{\infty} r^k\right]$

i.e. $\quad \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2r^2 \frac{d^2}{dr^2}\left[-1 - r + \frac{1}{1-r}\right] + 2r \frac{d}{dr}\left[-1 - r + \frac{1}{1-r}\right]$

i.e. $\quad \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2r^2 \left[0 + \frac{2}{(1-r)^3}\right] + 2r\left[-1 + \frac{1}{(1-r)^2}\right] = 2r\left[\frac{2r}{(1-r)^3} - 1 + \frac{1}{(1-r)^2}\right]$

$$\Rightarrow \quad -\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \ge 2r\left[-\frac{2r}{(1-r)^3} + 1 - \frac{1}{(1-r)^2}\right] = 2r\left[1 - \frac{1 - r + 2r}{(1-r)^3}\right] = 2r\left[1 - \frac{1+r}{(1-r)^3}\right]$$

Thus we have

$$\begin{aligned}
|\mathbf{g}(z_1) - \mathbf{g}(z_2)| &\ge r - \rho - \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \\
&\ge r - \rho + 2r\left[1 - \frac{1+r}{(1-r)^3}\right] = r + 2r - \frac{2r(1+r)}{(1-r)^3} - \rho = r\left[3 - \frac{2(1+r)}{(1-r)^3}\right] - \rho
\end{aligned}$$

Observe that

$$r\left[3-\frac{2(1+r)}{(1-r)^3}\right]-\rho>0 \quad \Leftrightarrow \quad r\left[3-\frac{2(1+r)}{(1-r)^3}\right]>\rho\geq 0$$

Consider

$$0\leq\rho<r\left[3-\frac{2(1+r)}{(1-r)^3}\right] \quad \Rightarrow \quad 3-\frac{2(1+r)}{(1-r)^3}>0 \quad \Rightarrow \quad 3>\frac{2(1+r)}{(1-r)^3}$$

$$\Rightarrow \quad 3(1-r)^3>2(1+r) \quad (0<r<1 \quad \Rightarrow \quad -r>-1 \quad \Rightarrow \quad 1-r>1-1=0)$$

$$\Rightarrow \quad 3[1-r^3-3r(1-r)]>2+2r \quad \Rightarrow \quad 3-3r^3-9r+9r^2>2+2r$$

$$\Rightarrow \quad 1-11r+9r^2-3r^3>0$$

$$\Rightarrow \quad 1-11\frac{1}{R}+9\frac{1}{R^2}-3\frac{1}{R^3}>0 \quad (rR=1, \quad \text{where} \quad R>1 \quad \text{since} \quad r<1)$$

$$\Rightarrow \quad R^3-11R^2+9R-3>0 \quad \Rightarrow \quad R^3-11R^2+9R>3 \quad \Rightarrow \quad R(R^2-11R+9)>3$$

$$\Rightarrow \quad R\left[R-\frac{-(-11)-\sqrt{(-11)^2-4(1)9}}{2}\right]\left[R-\frac{-(-11)+\sqrt{(-11)^2-4(1)9}}{2}\right]>3$$

$$\Rightarrow \quad R\left[R-\frac{11-\sqrt{121-36}}{2}\right]\left[R-\frac{11+\sqrt{121-36}}{2}\right]>3$$

$$\Rightarrow \quad R\left[R-\frac{11-\sqrt{121-36}}{2}\right]\left[R-\frac{11+\sqrt{121-36}}{2}\right]>3$$

$$\Rightarrow \quad R\left[R-\frac{11-\sqrt{85}}{2}\right]\left[R-\frac{11+\sqrt{85}}{2}\right]>3>0 \quad \left(\sqrt{85}\cong 9.2195\right)$$

$$\Rightarrow \quad \begin{cases} Either \quad R<\dfrac{11-\sqrt{85}}{2}\cong\dfrac{11-9.2195}{2}=\dfrac{2.2195}{2}=1.10975, \\ or \quad R>\dfrac{11+\sqrt{85}}{2}\cong\dfrac{11+9.2195}{2}=\dfrac{20.2195}{2}=10.10975 \end{cases}$$

Suppose that

$$1<R<\frac{11-\sqrt{85}}{2}<\frac{11+\sqrt{85}}{2} \quad i.e. \quad 1<R<1.10975<10.10975$$

$$\Rightarrow \quad 0<1.10975-R<1.10975-1=0.10975, \quad 0<10.10975-R<10.10975-1=9.10975$$

$$\Rightarrow \quad 0<[1.10975-R][10.10975-R]<(0.10975)(9.10975)=0.999795$$

$$\Rightarrow \quad 0<R[R-1.10975][R-10.10975]<1.10975(0.999795)=1.10952257<3$$

This is contradiction. Thus our supposition is wrong. Hence we have

$$R>\frac{11+\sqrt{85}}{2} \quad \Rightarrow \quad \frac{1}{r}>\frac{11+\sqrt{85}}{2} \quad \Rightarrow \quad r<\frac{2}{11+\sqrt{85}}=\frac{2\left(11-\sqrt{85}\right)}{121-85}=\frac{2\left(11-\sqrt{85}\right)}{36}$$

$$i.e. \quad 0\leq\rho=|z_1|\leq|z_2|=r<\frac{11-\sqrt{85}}{18}\cong 0.0989<1.$$

Thus we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)|\geq r\left[3-\frac{2(1+r)}{(1-r)^3}\right]-\rho>0$$

i.e. $\mathbf{g}(z_1) \neq \mathbf{g}(z_2) \Rightarrow \mathbf{g}(z)$ is univalent in the open disc $\{z/|z| < c < 1\}$.

Thus $\mathbf{g}(z) = z\mathbf{f}'(z) \in \mathbf{U} \Rightarrow |b_k| \leq k \quad (k = 2,3,4,\cdots)$ by BEIRBERBACH conjecture.

$$\Rightarrow \quad |k\,a_k| \leq k \quad (k = 2,3,4,\cdots) \quad \Rightarrow \quad |a_k| \leq 1 \quad (k = 2,3,4,\cdots).$$

Again consider

$$|\textstyle\sum_{k=2}^{n} ka_k[z_1^k - z_2^k]| \leq \sum_{k=2}^{n} k|a_k||z_1^k - z_2^k| \leq \sum_{k=2}^{n} k1[|z_1^k| + |z_2^k|]$$

$$= \textstyle\sum_{k=2}^{n} k[|z_1|^k + |z_2|^k] \leq \sum_{k=2}^{n} k[r^k + r^k].$$

i.e. $|\sum_{k=2}^{\infty} ka_k[z_1^k - z_2^k]| \leq \sum_{k=2}^{\infty} k\,2r^k = 2r\sum_{k=2}^{\infty} k\,r^{k-1}$

i.e. $|\sum_{k=2}^{\infty} ka_k[z_1^k - z_2^k]| \leq 2r \sum\limits_{k=2}^{\infty} \frac{d}{dr} r^k = 2r\frac{d}{dr}\sum\limits_{k=2}^{\infty} r^k = 2r\frac{d}{dr}\left[\sum\limits_{k=0}^{\infty} r^k - 1 - r\right] = 2r\left[\frac{1}{(1-r)^2} - 1\right]$

$$\Rightarrow \quad -|\textstyle\sum_{k=2}^{\infty} ka_k[z_1^k - z_2^k]| \geq 2r\left[1 - \dfrac{1}{(1-r)^2}\right]$$

Thus we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| \geq r - \rho - |\textstyle\sum_{k=2}^{\infty} ka_k[z_1^k - z_2^k]|$$

$$\geq r - \rho + 2r\left[1 - \frac{1}{(1-r)^2}\right] = r + 2r - \frac{2r}{(1-r)^2} - \rho = r\left[3 - \frac{2}{(1-r)^3}\right] - \rho$$

Observe that

$$r\left[3 - \frac{2}{(1-r)^2}\right] - \rho > 0 \quad \Leftrightarrow \quad r\left[3 - \frac{2}{(1-r)^2}\right] > \rho \geq 0$$

Consider

$$0 \leq \rho < r\left[3 - \frac{2}{(1-r)^2}\right] \quad \Rightarrow \quad 3 - \frac{2}{(1-r)^2} > 0 \quad \Rightarrow \quad 3 > \frac{2}{(1-r)^2}$$

$$\Rightarrow \quad (1-r)^2 > \frac{2}{3} \quad (0 < r < 1 \quad \Rightarrow \quad -r > -1 \quad \Rightarrow \quad 1 - r > 1 - 1 = 0)$$

$$\Rightarrow \quad 1 - r > \frac{\sqrt{2}}{\sqrt{3}} \quad \Rightarrow \quad 1 - \frac{\sqrt{2}}{\sqrt{3}} > r \quad \Rightarrow \quad r < 1 - \frac{\sqrt{2}}{\sqrt{3}} = 1 - \frac{\sqrt{6}}{3} = 0.1835 < 1$$

Thus, for $0 \leq \rho = |z_1| \leq |z_2| = r < c < 1,$ we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| \geq r\left[3 - \frac{2}{(1-r)^3}\right] - \rho > 0 \qquad c = 1 - \frac{\sqrt{6}}{3} = \frac{3 - \sqrt{6}}{3}$$

Let $\mathbf{f}(z)$ is cap like function.

$$\Rightarrow \begin{cases} 0 < \mathbf{Re}\left[1 + \dfrac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] = \mathbf{Re}\left[\dfrac{\mathbf{f}'(z) + z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] = \mathbf{Re}\left[\dfrac{1}{\mathbf{f}'(z)}\dfrac{d}{dz}[z\mathbf{f}'(z)]\right] = \mathbf{Re}\left[\dfrac{1}{z\mathbf{f}'(z)}z\dfrac{d}{dz}[z\mathbf{f}'(z)]\right], \\ \text{and } \mathbf{f}(z) \text{ is analytic in the open unit disc } \{z/|z| < 1\} \text{ with } \mathbf{f}(0) = 0, \mathbf{f}'(0) = 1. \end{cases}$$

$\Rightarrow \quad \mathbf{g}'(z) = z\mathbf{f}''(z) + \mathbf{f}'(z)$ will exist since $\mathbf{f}(z)$ is analytic, and $\mathbf{g}(0) = 0\mathbf{f}'(0) = 0 \times 1 = 0$

$\Rightarrow \quad \mathbf{g}'(0) = 0\mathbf{f}''(0) + \mathbf{f}'(0) = \mathbf{f}'(0) = 1$ and $\mathbf{g}(z)$ is analytic in the open unit disc $\{z/|z| < 1\}$.

Hence $\mathbf{g}(z) = z\mathbf{f}'(z)$ is star like function is in the disc $|z| < c < 1.//$

Problemem : Let $\mathbf{f}(z)$ is star like function. Then

$$0 < \mathbf{Re}\left[\frac{1}{z\mathbf{f}(z)} z \frac{d}{dz}[z\mathbf{f}(z)]\right]$$

and $z\mathbf{f}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$, but $z\mathbf{f}(z)$ is not univalent in the open disc $\{z/|z|<r\le 1\}$, thus $z\mathbf{f}(z)$ is not star like function.

Proof : Let $\mathbf{f}(z)$ is star like function.

$$\Rightarrow \quad 0 < \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right] < \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]+1 = \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+1\right] = \mathbf{Re}\left[\frac{z\mathbf{f}'(z)+\mathbf{f}(z)}{\mathbf{f}(z)}\right] = \mathbf{Re}\left[\frac{1}{\mathbf{f}(z)}\frac{d}{dz}[z\mathbf{f}(z)]\right]$$

$\mathbf{f}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$ with $\mathbf{f}(0)=0,\ \mathbf{f}'(0)=1,$
and $\mathbf{f}(z)$ is univalent in an open disc $\{z/|z|<r<1\}$

Put $\mathbf{g}(z) = z\mathbf{f}(z)$

$\Rightarrow \quad \mathbf{g}'(z) = z\mathbf{f}'(z)+\mathbf{f}(z)$ will exist since $\mathbf{f}(z)$ is analytic, and $\mathbf{g}(0) = 0\mathbf{f}(0) = 0\times 0 = 0$

$\Rightarrow \quad \mathbf{g}'(0) = 0\mathbf{f}'(0)+\mathbf{f}(0) = \mathbf{f}'(0) = 0$ and $\mathbf{g}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$.

But $\mathbf{g}(z) = z\mathbf{f}(z)$ is not star like function since $\mathbf{g}'(0)\ne 1$.

Taylor series of $\mathbf{f}(z)$ about the origin is $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} a_k z^k$

$\Rightarrow \quad z\mathbf{f}(z) = z[z+\sum_{k=2}^{\infty} a_k z^k] = z^2 + \sum_{k=2}^{\infty} a_k z^{k+1}$

Consider

$$\mathbf{g}(z_1)-\mathbf{g}(z_2) = z_1\mathbf{f}(z_1)-z_2\mathbf{f}(z_2) = [z_1^2+\sum_{k=2}^{\infty} a_k z_1^{k+1}] - [z_2^2+\sum_{k=2}^{\infty} a_k z_2^{k+1}]$$

$$= z_1^2 - z_2^2 + \sum_{k=2}^{\infty} a_k z_1^{k+1} - \sum_{k=2}^{\infty} a_k z_2^{k+1} = z_1^2 - z_2^2 + \sum_{k=2}^{\infty} a_k [z_1^{k+1} - z_2^{k+1}]$$

Let $z_1 \ne z_2$ in the open disc $\{z/|z|<r\le 1\}$ such that $z_1 = -z_2 \quad \Rightarrow \quad z_1^2 = z_2^2$

$\Rightarrow \quad \mathbf{g}(z_1)-\mathbf{g}(z_2) = \sum_{k=2}^{\infty} a_k z_1^{k+1} - \sum_{k=2}^{\infty} a_k z_2^{k+1}$ may or may not be 0.

$\Rightarrow \quad \mathbf{g}(z)$ is not univalent in the open disc $\{z/|z|<r\le 1\}$. / /

Definition 5 : A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z|<1\}$ with $\mathbf{f}(0)=0,$ $\mathbf{f}'(0)=1$ is said to be *cap like function of order* $\alpha(c)$ if

$$\mathbf{Re}\left[1+\frac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] > \alpha(c), \quad |z|<c\le 1, \quad 0\le\alpha(c)<1.$$

Definition 6 : A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z|<1\}$ and univalent in open disc $\{z/|z|<c\le 1\}$ with $\mathbf{f}(0)=0,\ \mathbf{f}'(0)=1$ is said to be *star like function of order* $\alpha(c)$ if

$$\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right] > \alpha(c), \quad |z|<c\le 1, \quad 0\le\alpha(c)<1.$$

Theorem 3 : If $\mathbf{f}(z)$ is star like function of order $\alpha = 1$, then $(z+1)\mathbf{f}(z)$ is star like function of order $\alpha = c \le 0.5$

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Proof : Let $\mathbf{f}(z)$ is star like function of order $\alpha = 1$.

$$\Rightarrow \begin{cases} \mathbf{Re}\left[\dfrac{z\,\mathbf{f}'(z)}{\mathbf{f}(z)}\right] > 1, \\ \mathbf{f}(z) \text{ is analytic in the open unit disc } \{z/|z|<1\} \text{ with } \mathbf{f}(0)=0,\ \mathbf{f}'(0)=1 \\ \text{and } \mathbf{f}(z) \text{ is univalent in the open disc } \{z/|z|<r\le 1\} \end{cases}.$$

Put $\mathbf{g}(z) = (z+1)\mathbf{f}(z) \quad \Rightarrow$

$$\Rightarrow \begin{cases} \mathbf{g}'(z) = (z+1)\mathbf{f}'(z) + \mathbf{f}(z) \text{ will exist since } \mathbf{f}(z) \text{ is analytic,} \\ \text{and } \mathbf{g}(0) = 1\,\mathbf{f}(0) = 1\times 0 = 0, \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{g}'(0) = (0+1)\mathbf{f}'(0) + \mathbf{f}(0) = \mathbf{f}'(0) + 0 = 1, \\ \text{and } \mathbf{g}(z) \text{ is analytic in the open unit disc } \{z/|z|<1\}. \end{cases}$$

Taylor series of $\mathbf{f}(z)$ about the origin is $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} a_k z^k$

$$\Rightarrow \quad z\,\mathbf{f}(z) = z\left[z + \sum_{k=2}^{\infty} a_k z^k\right] = z^2 + \sum_{k=2}^{\infty} a_k z^{k+1}$$

$$\Rightarrow \quad (z+1)\mathbf{f}(z) = z^2 + \sum_{k=2}^{\infty} a_k z^{k+1} + z + \sum_{k=2}^{\infty} a_k z^k$$

$$= z + z^2 + \sum_{k-1=2}^{\infty} a_{k-1} z^{k-1+1} + \sum_{k=2}^{\infty} a_k z^k$$

$$= z + z^2 + \sum_{k=3}^{\infty} a_{k-1} z^k + a_2 + \sum_{k=3}^{\infty} a_k z^k = z + (1+a_2)z^2 + \sum_{k=3}^{\infty} (a_{k-1} + a_k) z^k$$

Put $1 + a_2 = b_2$, $a_{k-1} + a_k = b_k \quad (k = 3,4,5,\cdots)$.

Thus $\mathbf{g}(z) = (z+1)\mathbf{f}(z) = z + \sum_{k=2}^{\infty} b_k z^k$.

Let $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0 \quad \Rightarrow \quad |z_1 - z_2| \ne 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| \le |z_1 - z_2|$ by triangle inequality.

Consider

$$\mathbf{g}(z_1) - \mathbf{g}(z_2) = (z_1+1)\mathbf{f}'(z_1) - (z_2+1)\mathbf{f}'(z_2)$$

$$= \left[z_1 + \sum_{k=2}^{\infty} b_k z_1^k\right] - \left[z_2 + \sum_{k=2}^{\infty} b_k z_2^k\right]$$

$$= z_1 - z_2 + \left[\sum_{k=2}^{\infty} b_k z_1^k - \sum_{k=2}^{\infty} b_k z_2^k\right] = z_1 - z_2 + \sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right|$

$$\Rightarrow \quad |\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right| \ge r - \rho - \left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right|$$

Consider

$$\left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{\infty} |b_k|\left[|z_1|^k + |z_2|^k\right] < \sum_{k=2}^{\infty} |b_k|\left[r^k + r^k\right] = 2\sum_{k=2}^{\infty} |b_k|\, r^k.$$

Note that $\mathbf{f}(z) \in \mathbf{U} \quad \Rightarrow \quad |a_k| \le 1 \quad (k = 2,3,4,\cdots)$ by Theorem **2.** Then

$$|b_2| = |1 + a_2| \le 1 + |a_2| \le 1 + 1 = 2, \qquad |b_k| = |a_{k-1} + a_k| \le |a_{k-1}| + |a_k| \le 1 + 1 = 2.$$

Thus we have

$$\left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right| \le 2\sum_{k=2}^{\infty} |b_k|\, r^k \le 2\sum_{k=2}^{\infty} 2r^k = 4\sum_{k=2}^{\infty} r^k$$

Thus we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} b_k\left[z_1^k - z_2^k\right]\right| \ge r - \rho - 4\sum_{k=2}^{\infty} r^k$$

i.e. $|\mathbf{g}(z_1)-\mathbf{g}(z_2)| \geq r-\rho-4\frac{r^2}{1-r} = r\left[1-4\frac{r}{1-r}\right]-\rho$

Observe that

$$r\left[1-4\frac{r}{1-r}\right]-\rho>0 \quad \Leftrightarrow \quad r\left[1-4\frac{r}{1-r}\right]>\rho\geq 0$$

Consider

$$0\leq\rho<r\left[1-4\frac{r}{1-r}\right] \quad \Rightarrow \quad 1-4\frac{r}{1-r}>0 \quad \Rightarrow \quad 1>4\frac{r}{1-r}$$

$$\Rightarrow \quad 1-r>4r \quad (0<r<1 \quad \Rightarrow \quad -r>-1 \quad \Rightarrow \quad 1-r>1-1=0)$$

$$\Rightarrow \quad 1>5r \quad \Rightarrow \quad 5r<1 \quad \Rightarrow \quad r<0.2$$

Thus, for $0\leq\rho=|z_1|\leq|z_2|=r<0.2$, we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| \geq r-\rho-4\frac{r^2}{1-r} = r\left[1-4\frac{r}{1-r}\right]-\rho>0 \quad \Rightarrow \quad \mathbf{g}(z_1)\neq\mathbf{g}(z_2)$$

Thus $\mathbf{g}(z)$ is univalent in the open disc $\{z/|z|<0.2<1\}$.

We have

$$\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)] = \frac{z[(z+1)\mathbf{f}'(z)+\mathbf{f}(z)]}{(z+1)\mathbf{f}(z)} = \frac{(z+1)z\mathbf{f}'(z)+z\mathbf{f}(z)}{(z+1)\mathbf{f}(z)} = \frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+\frac{z}{z+1}$$

Consider

$$\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+\frac{z}{z+1}\right] = \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]+\mathbf{Re}\left[\frac{z}{z+1}\right] > 1+\mathbf{Re}\left[\frac{z}{z+1}\right] = 1+\mathbf{Re}\left[\frac{z+1-1}{z+1}\right]$$

i.e. $$\mathbf{Re}\left[\frac{z}{(z+1)\mathbf{f}(z)}\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > 1+\mathbf{Re}\left[1-\frac{1}{z+1}\right] = 1+1-\mathbf{Re}\left[\frac{1}{z+1}\right]$$

i.e. $$\mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > 2-\mathbf{Re}\left[\frac{1}{x+\iota y+1}\right] = 2-\mathbf{Re}\left[\frac{1+x-\iota y}{(1+x)^2+y^2}\right]$$

i.e. $$\mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > 2-\frac{1+x}{(1+x)^2+y^2}$$

Let $|z|<c<1 \quad \Rightarrow \quad |z|^2<c^2<1 \quad \Rightarrow \quad x^2+y^2<c^2<1$

$$\Rightarrow \quad x^2<x^2+y^2<c^2<1 \quad \Rightarrow \quad -1<-c<x<c<1 \quad \Rightarrow \quad 0<1-c<1+x<1+c<2$$

But $\quad 0<(1+x)^2<(1+x)^2+y^2$

$$\Rightarrow \quad \frac{1}{(1+x)^2+y^2}<\frac{1}{(1+x)^2} \quad \Rightarrow \quad \frac{1+x}{(1+x)^2+y^2}<\frac{1+x}{(1+x)^2}=\frac{1}{1+x}<\frac{1}{1-c}$$

$$\Rightarrow \quad -\frac{1+x}{(1+x)^2+y^2}>-\frac{1}{1-c} \quad \Rightarrow \quad 2-\frac{1+x}{(1+x)^2+y^2}>2-\frac{1}{1-c}=\frac{2-2c-1}{1-c}=\frac{1-2c}{1-c}$$

Thus we have

$$\mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > 2-\frac{1+x}{(1+x)^2+y^2} > \frac{1-2c}{1-c}\geq 0.$$

for $1-2c\geq 0$ or $1\geq 2c$ or $2c\leq 1$ or $c\leq 0.5<1$.

$$c = 0.2 < 0.5 < 1 \qquad \Rightarrow \qquad \frac{1-2c}{1-c} = \frac{1-2(0.2)}{1-0.2} = \frac{0.6}{0.8} = \frac{3}{4}$$

Hence $\mathbf{g}(z) = (z+1)\mathbf{f}(z)$ is star like function of order $\alpha = 4^{-1}3$

in the open disc $|z| < 0.2 < 1.//$

Definition 6 : *Hadamard product* (*Convolution*) of two analytic functions $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k$ in the open disc $|z| < r_1$ and $\mathbf{g}(z) = \sum_{k=0}^{\infty} b_k z^k$ in the open disc $|z| < r_2$ is denoted by $\mathbf{f} * \mathbf{g}$ and is defined as an anlytic function $(\mathbf{f} * \mathbf{g})(z) = \sum_{k=0}^{\infty} a_k b_k z^k$ in the open disc $|z| < r_1 r_2$.

Theorem 4 : Let $\mathbf{L}(z) = z(1-z)^{-1}$. Then $s_n(z,\mathbf{L})$ $(n = 2,3,4,\cdots)$ is cap like function in disc $|z| < 0.25$

Proof : $\mathbf{L}(z) = z(1-z)^{-1} = z\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k+1} = \sum_{k-1=0}^{\infty} z^{k-1+1} = \sum_{k=1}^{\infty} z^k = z + \sum_{k=2}^{\infty} a_k z^k$

where $a_k = 1$ $(k = 2,3,4,\cdots)$. Then we have

$$s_n(z,\mathbf{L}) = z + \sum_{k=2}^{n} z^k = \sum_{k=1}^{n} z^k = \sum_{k+1=1}^{n} z^{k+1} = z\sum_{k=0}^{n-1} z^{k+1} = z\frac{1-z^n}{1-z}$$

$$\Rightarrow \qquad s_n'(z,\mathbf{L}) = \frac{(1-z)[1-(n+1)z^n] - (0-1)[z - z^{n+1}]}{(1-z)^2}$$

$$= \frac{1 - z - (n+1)z^n + (n+1)z^{n+1} + z - z^{n+1}}{(1-z)^2} = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}$$

$$\Rightarrow \qquad \log s_n'(z,\mathbf{L}) = \log\left[1-(n+1)z^n + nz^{n+1}\right] - 2\log(1-z)$$

By taking the derivative on both sides, we have

$$\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})} = \frac{0-(n+1)nz^{n-1} + n(n+1)z^n}{1-(n+1)z^n + nz^{n+1}} - 2\frac{-1}{1-z} = \frac{(n+1)nz^{n-1}[-1+z]}{1-(n+1)z^n + nz^{n+1}} + \frac{2}{1-z}$$

$$\Rightarrow \qquad z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})} = \frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n + nz^{n+1}} + \frac{2z}{1-z} = \frac{N(z)}{D(z)} + \frac{2z}{1-z}$$

$$\Rightarrow \qquad 1 + z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})} = 1 + \frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n + nz^{n+1}} + \frac{2z}{1-z} = \frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n + nz^{n+1}} + \frac{1+z}{1-z}$$

To simplify the notations, put

$$N(z) = (n+1)nz^n[-1+z], \qquad D(z) = 1-(n+1)z^n + nz^{n+1}, \qquad \frac{1+z}{1-z} = w = u + \iota v$$

$$\Rightarrow \qquad 1 + z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})} = \frac{N(z)}{D(z)} + w$$

$$\Rightarrow \qquad \mathbf{Re}\left[1 + z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}\right] = \mathbf{Re}\left[\frac{N(z)}{D(z)} + w\right] = \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] + \mathbf{Re}\, w = \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] + u \qquad \cdots\cdots\cdots\cdots(1)$$

We have

$$w = \frac{1+z}{1-z} \quad \Leftrightarrow \quad w - wz = 1 + z \quad \Leftrightarrow \quad w - 1 = z + wz \quad \Leftrightarrow \quad w - 1 = (1+w)z$$

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Consider

$$|z|=\frac{1}{4} \quad \Leftrightarrow \quad \left|\frac{w-1}{w+1}\right|=\frac{1}{4} \quad \Leftrightarrow \quad 4|w-1|=|w+1|$$

$$\Leftrightarrow \quad 4|u+\imath v-1|=|u+\imath v+1| \quad \Leftrightarrow \quad 16|u-1+\imath v|^2=|u+1+\imath v|^2$$

$$\Leftrightarrow \quad 16[(u-1)^2+v^2]=[(u+1)^2+v^2] \quad \Leftrightarrow \quad 16[u^2-2u+1+v^2]=[u^2+2u+1+v^2]$$

$$\Leftrightarrow \quad 16u^2-32u+16+16v^2=u^2+2u+1+v^2 \quad \Leftrightarrow \quad 15u^2-34u+15+15v^2=0$$

$$\Leftrightarrow \quad u^2-\frac{34}{15}u+1+v^2=0 \quad \Leftrightarrow \quad u^2-2\frac{17}{15}u+\left(\frac{17}{15}\right)^2-\left(\frac{17}{15}\right)^2+1+v^2=0$$

$$\Leftrightarrow \quad \left(u-\frac{17}{15}\right)^2+v^2=\frac{289}{225}-1=\frac{289-225}{225}=\frac{64}{225}=\left(\frac{8}{15}\right)^2.$$

$$\mathbf{max}\left(u-\frac{17}{15}\right)^2=\mathbf{max}\left[\left(\frac{8}{15}\right)^2-v^2\right]=\left(\frac{8}{15}\right)^2 \qquad i.e. \qquad \mathbf{max} \text{ will exist at } v=0$$

$$\Rightarrow \quad \left(u-\frac{17}{15}\right)^2=\left(\frac{8}{15}\right)^2 \quad \Rightarrow \quad u-\frac{17}{15}=\pm\frac{8}{15} \quad \Rightarrow \quad u=\frac{17}{15}\pm\frac{8}{15}$$

$$\Rightarrow \quad u=\frac{17}{15}+\frac{8}{15}=\frac{25}{15} \quad \text{or} \quad u=\frac{17}{15}-\frac{8}{15}=\frac{9}{15}.$$

Hence it is clear that the Mobius **(**Bilinear**)** transformation

$$w=\frac{1+z}{1-z}$$

maps the circle $|z|=4^{-1}$ in xy-plane into the the circle

$$\left(u-\frac{17}{15}\right)^2+v^2=\left(\frac{8}{15}\right)^2$$

in uv-plane such that the line segment AB on u-axis $(v=0)$ is a diameter where

$$A=\left(\frac{9}{15},0\right)=\left(\frac{3}{5},0\right), \quad \text{and} \quad B=\left(\frac{25}{15},0\right)=\left(\frac{5}{3},0\right).$$

$$\Rightarrow \quad \frac{9}{15}\le u\le\frac{25}{15} \quad \text{or} \quad \frac{3}{5}\le u\le\frac{5}{3} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2).$$

Observe that $|N(z)|=|(n+1)nz^n[-1+z]|=(n+1)n|z^n|\,|-1+z|\le(n+1)n|z|^n\,[1+|z|]$

$$\Rightarrow \quad |N(z)|\le(n+1)n|z|^n[1+|z|]\le(n+1)n\,(4^{-1})^n\,[1+4^{-1}]=(n+1)n\,4^{-n-1}[4+1]$$

$$\Rightarrow \quad |N(z)|\le5(n+1)n\,4^{-n-1} \qquad \text{for} \quad |z|=4^{-1}$$

Consider $\quad |nz^{n+1}-(n+1)z^n|\le|nz^{n+1}|+|-(n+1)z^n|=n|z|^{n+1}+(n+1)|z|^n$

Put $|z|=4^{-1} \quad \Rightarrow \quad |nz^{n+1}-(n+1)z^n|\le n4^{-n-1}+(n+1)4^{-n}<1$

$$\Rightarrow \quad -|nz^{n+1}-(n+1)z^n|\ge-n4^{-n-1}-(n+1)4^{-n}>-1$$

$$\Rightarrow \quad 1-|nz^{n+1}-(n+1)z^n|\ge1-n4^{-n-1}-(n+1)4^{-n}>1-1=0$$

But $\quad |D(z)|=|1+nz^{n+1}-(n+1)z^n|\ge1-|nz^{n+1}-(n+1)z^n|\ge0$

$$\Rightarrow \quad \frac{1}{|D(z)|}\le\frac{1}{1-n4^{-n-1}-(n+1)4^{-n}} \qquad \text{for} \quad |z|=4^{-1}.$$

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Thus we have, for $|z| = 4^{-1}$,

$$\left|\frac{N(z)}{D(z)}\right| = \frac{|N(z)|}{|D(z)|} \le \frac{5(n+1)n\,4^{-n-1}}{1 - n4^{-n-1} - (n+1)4^{-n}} = \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3).$$

Observe that

$$n = 2 \quad \Rightarrow \quad \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} = \frac{5(2+1)2}{4^{2+1} - 2 - (2+1)4} = \frac{10(3)}{64 - 2 - 12} = \frac{10(3)}{50} = \frac{3}{5}.$$

$$\frac{5(n+1)n}{4^{n+1} - n - (n+1)4} \le \frac{3}{5} \quad \Leftrightarrow \quad \frac{25}{12} \le \frac{4^{n+1} - n - (n+1)4}{4(n+1)n} = \frac{4^n}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \quad \cdots\cdots\cdots(4).$$

$$\frac{1}{4(n+1)} < 1, \quad \frac{1}{n} < 1 \quad \Rightarrow \quad -\frac{1}{4(n+1)} > -1, \quad -\frac{1}{n} > -1 \quad \Rightarrow \quad -\frac{1}{4(n+1)} - \frac{1}{n} > -1 - 1$$

for any $n = 2, 3, 4, \cdots$

$$n = 3 \quad \Rightarrow \quad \frac{4^n}{n(n+1)} = \frac{4^3}{3(3+1)} = \frac{64}{12} > \frac{25}{12}$$

Observe that for all integers k

$$\begin{cases} \dfrac{4^{k+1}}{(k+1)(k+2)} > \dfrac{4^k}{k(k+1)} \quad \Leftrightarrow \quad \dfrac{4^k 4}{(k+1)(k+2)} > \dfrac{4^k}{k(k+1)} \quad \Leftrightarrow \quad \dfrac{4}{k+2} > \dfrac{1}{k} \\ \Leftrightarrow \quad 4k > k+2 \quad \Leftrightarrow \quad 3k > 2 \end{cases}$$

Since $3k > 2$ for all integers $k \ge 1$, we have

$$\frac{4^{k+1}}{(k+1)(k+2)} > \frac{4^k}{k(k+1)} > \cdots > \frac{4^3}{3(3+1)} = \frac{64}{12}.$$

Thus we have

$$\frac{4^n}{n(n+1)} \ge \frac{64}{12} \quad (n = 3, 4, 5, \cdots) \quad \Rightarrow \quad \frac{4^n}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \ge \frac{64}{12} - 2 = \frac{40}{12} > \frac{25}{12}$$

Thus, from (4), we have

$$\frac{5(n+1)n}{4^{n+1} - n - (n+1)4} \le \frac{3}{5}$$

Thus, from (3), we have

$$\left|\frac{N(z)}{D(z)}\right| \le \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} \le \frac{3}{5} \quad \Rightarrow \quad -\frac{3}{5} \le -\left|\frac{N(z)}{D(z)}\right|.$$

We know that $|\mathbf{f}(z)|^2 = [\mathbf{Re}\,\mathbf{f}(z)]^2 + [\mathbf{Im}\,\mathbf{f}(z)]^2 \ge [\mathbf{Re}\,\mathbf{f}(z)]^2$

$$\Rightarrow \quad [\mathbf{Re}\,\mathbf{f}(z)]^2 \le |\mathbf{f}(z)|^2 \quad \Rightarrow \quad -|\mathbf{f}(z)| \le \mathbf{Re}\,\mathbf{f}(z) \le |\mathbf{f}(z)|.$$

Hence we have

$$-\frac{3}{5} \le -\left|\frac{N(z)}{D(z)}\right| < \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] \quad \Rightarrow \quad \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] > -\frac{3}{5}.$$

Therefore, from (1) and (2), we have

$$\mathbf{Re}\left[1 + z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}\right] = \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] + u > -\frac{3}{5} + \frac{3}{5} = 0.$$

It remains to show that 0.25 is maximal radius. This is seen for $s_2(z,\mathbf{L}) = z + z^2$. Then

$$1 + z\frac{s_2''(z,\mathbf{L})}{s_2'(z,\mathbf{L})} = 1 + z\frac{0+2}{1+2z} = \frac{1+4z}{1+2z}$$

has singularity at $z = -0.25$, and thus analytic within $|z| < 0.25$.

Clearly $s_n(z,\mathbf{L}) = z + \sum_{k=2}^{n} z^k$ is analytic within $|z| < 0.25$, and $s_n(0,\mathbf{L}) = 0$,

Since $s_n'(z,\mathbf{L}) = 1 + \sum_{k=2}^{n} k\,z^{k-1}$, we have $s_n'(0,\mathbf{L}) = 1 + \sum_{k=2}^{n} k\,0^{k-1} = 1$.

Hence $s_n(z,\mathbf{L})$ is cap like function in the open disc $|z| < 0.25$.//

Theorem 5 : Let $\mathbf{f}(z)$, $\mathbf{g}(z)$ are cap like functions and are univalent. Then $(\mathbf{f}*\mathbf{g})(z)$ is cap like function and is univalent.

Proof : Let $\mathbf{f}(z)$, $\mathbf{g}(z)$ are cap like functions. Then $\mathbf{f}(z)$, $\mathbf{g}(z)$ are analytic functions in the unit open disc $|z| < 1$ with donditions $\mathbf{f}(0) = 0 = \mathbf{g}(0)$ and $\mathbf{f}'(0) = 1 = \mathbf{g}'(0)$.

Note that $\{\mathbf{f}(z), \mathbf{g}(z)\} \subset \mathbf{U}$ since $\mathbf{f}(z)$, $\mathbf{g}(z)$ are univalent in the unit open disc $|z| < 1$.

$\Rightarrow$ By Theorem 2, $|a_k| \le 1$ $(k = 2,3,4,\cdots)$ and $|b_k| \le 1$ $(k = 2,3,4,\cdots)$.

Taylor's expansion of $\mathbf{f}(z)$, $\mathbf{g}(z)$ about $z = 0$ is given by

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \sum_{k=1}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$\mathbf{g}(z) = \sum_{k=0}^{\infty} b_k z^k = b_0 + b_1 z + \sum_{k=1}^{\infty} b_k z^k = z + \sum_{k=2}^{\infty} b_k z^k.$$

$$\Rightarrow \quad (\mathbf{f}*\mathbf{g})(z) = \sum_{k=0}^{\infty} a_k b_k z^k = a_0 b_0 + a_1 b_1 z + \sum_{k=1}^{\infty} a_k b_k z^k = z + \sum_{k=2}^{\infty} a_k b_k z^k$$

Clearly $(\mathbf{f}*\mathbf{g})(z)$ is analytic function in the unit open disc $|z| < 1$ with conditions $(\mathbf{f}*\mathbf{g})(0) = 0$ and $(\mathbf{f}*\mathbf{g})'(0) = 1$.

Let us consider

$$\mathbf{Re}\left[1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] = 1 + \mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right].$$

We know that

$$\mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] \le \left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right| \quad \Rightarrow \quad -\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right| \le \mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] \le \left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|$$

By Triangle inequality,

$$1 - \left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right| \le \left|1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right| = \mathbf{Re}\left[1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]$$

$$\Rightarrow \quad \mathbf{Re}\left[1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] \ge 1 - \left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right| = 1 - \frac{|z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|} = \frac{|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|}$$

We have $\quad (\mathbf{f}*\mathbf{g})'(z) = 1 + \sum_{k=2}^{\infty} a_k b_k k z^{k-1}$

$$\Rightarrow \quad (\mathbf{f}*\mathbf{g})''(z) = \sum_{k=2}^{\infty} a_k b_k k(k-1) z^{k-2} \quad \Rightarrow \quad z(\mathbf{f}*\mathbf{g})''(z) = \sum_{k=2}^{\infty} a_k b_k k(k-1) z^{k-1}$$

$$\Rightarrow \quad |z(\mathbf{f}*\mathbf{g})''(z)| = \left|\sum_{k=2}^{\infty} a_k b_k k(k-1) z^{k-1}\right|$$

$$\le \sum_{k=2}^{\infty} |a_k||b_k|\, k(k-1)|z|^{k-1} \le \sum_{k=2}^{\infty} 1\,k(k-1)|z|^{k-1}$$

Put $|z| = r \quad \Rightarrow \quad |z(\mathbf{f}*\mathbf{g})''(z)| \le \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1}$.

By Triangle inequality, $\quad 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \le |1 + \sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| = |(\mathbf{f}*\mathbf{g})'(z)|$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| \ge 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}|$

But $\quad |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \le \sum_{k=2}^{\infty} |a_k||b_k| k |z|^{k-1} \le \sum_{k=2}^{\infty} 1k|z|^{k-1} = \sum_{k=2}^{\infty} k\, r^{k-1}$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| \ge 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \ge 1 - \sum_{k=2}^{\infty} k\, r^{k-1}$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 1 - \sum_{k=2}^{\infty} k\, r^{k-1} - \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1} = 1 - \sum_{k=2}^{\infty} k^2 r^{k-1}$.

We have

$$\sum_{k=2}^{\infty} k^2 r^{k-1} = \sum_{k=2}^{\infty}[k(k-1)+k] r^{k-1} = \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1}$$

$$= \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1} = r\sum_{k=2}^{\infty} k(k-1) r^{k-2} + \sum_{k=2}^{\infty} k\, r^{k-1}.$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = r\sum_{k=2}^{\infty}\frac{d^2}{dr^2} r^k + \sum_{k=2}^{\infty}\frac{d}{dr} r^k = r\frac{d^2}{dr^2}\sum_{k=2}^{\infty} r^k + \frac{d}{dr}\sum_{k=2}^{\infty} r^k = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\sum_{k=2}^{\infty} r^k$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\left[\sum_{k=0}^{\infty} r^k - 1 - r\right] = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\left[\frac{1}{1-r} - 1 - r\right]$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = r\frac{2}{(1-r)^3} + \frac{1}{(1-r)^2} - 1 = \frac{2r+1-r}{(1-r)^3} - 1 = \frac{r+1}{(1-r)^3} - 1$$

Thus we have

$$|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 1 - \sum_{k=2}^{\infty} k^2 r^{k-1} = 1 - \frac{r+1}{(1-r)^3} + 1 = 2 - \frac{r+1}{(1-r)^3}$$

We have

$$2 - \frac{r+1}{(1-r)^3} > 0 \quad \Leftrightarrow \quad 2 > \frac{r+1}{(1-r)^3} \quad \Leftrightarrow \quad 2(1-r)^3 > r+1$$

$$\Leftrightarrow \quad 2[1 - r^3 - 3r(1-r)] > r+1 \quad \Leftrightarrow \quad 2 - 2r^3 - 6r + 6r^2 > r+1$$

$$\Leftrightarrow \quad 1 - 7r + 6r^2 - 2r^3 > 0 \quad \Leftrightarrow \quad 1 - 7\frac{1}{R} + 6\frac{1}{R^2} - 2\frac{1}{R^3} > 0 \quad (Rr = 1,\ \ R > 1 \ \ \text{since} \ \ r < 1)$$

$$\Leftrightarrow \quad R^3 - 7R^2 + 6R - 2 > 0 \quad \Leftrightarrow \quad R^3 - 7R^2 + 6R > 2 \quad \Leftrightarrow \quad R(R^2 - 7R + 6) > 2$$

$$\Leftrightarrow \quad R(R-1)(R-6) > 2$$

Let us consider $\quad R(R-1)(R-6) > 2$

$$\Rightarrow \quad R > 6 \quad (\because R > 1) \quad \Rightarrow \quad \frac{1}{r} > 6 \quad \Rightarrow \quad r < \frac{1}{6} \quad \Rightarrow \quad |z| < \frac{1}{6} < 1$$

Thus, for $|z| < 6^{-1} < 1$, we have

$$|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 2 - \frac{r+1}{(1-r)^3} > 0$$

$$\Rightarrow \quad \mathbf{Re}\left[1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] \ge \frac{|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|} > 0.$$

Hence $(\mathbf{f}*\mathbf{g})(z)$ is cap like function in the open disc $|z| < 6^{-1} < 1$.

Let $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0 \quad \Rightarrow \quad |z_1 - z_2| \ne 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

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Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| \le |z_1 - z_2|$ by triangle inequality.

Consider

$$(\mathbf{f} * \mathbf{g})(z_1) - (\mathbf{f} * \mathbf{g})(z_2) = z_1(\mathbf{f} * \mathbf{g})'(z_1) - z_2(\mathbf{f} * \mathbf{g})'(z_2)$$
$$= \left[z_1 + \sum_{k=2}^{\infty} a_k b_k k z_1^k\right] - \left[z_2 + \sum_{k=2}^{\infty} k a_k b_k z_2^k\right]$$
$$= z_1 - z_2 + \left[\sum_{k=2}^{\infty} k a_k b_k z_1^k - \sum_{k=2}^{\infty} k a_k b_k z_2^k\right] = z_1 - z_2 + \sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]\right|$

$$\Rightarrow \quad |(\mathbf{f} * \mathbf{g})(z_1) - (\mathbf{f} * \mathbf{g})(z_2)| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]\right| \ge r - \rho - \left|\sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]\right|$$

Again by triangle inequality; and since $|a_k| \le 1$, $|b_k| \le 1$; we have

$$\left|\sum_{k=2}^{n} k a_k b_k \left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{n} \left|k a_k b_k \left[z_1^k - z_2^k\right]\right| = \sum_{k=2}^{n} k |a_k| |b_k| |z_1^k - z_2^k|$$
$$\le \sum_{k=2}^{n} 1(1) \left[|z_1^k| + |z_2^k|\right]$$
$$= \sum_{k=2}^{n} \left[|z_1|^k + |z_2|^k\right] \le \sum_{k=2}^{n} \left[r^k + r^k\right]$$

since $|z_1| \le |z_2| = r$.

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \le 2\sum_{k=2}^{\infty} r^k = 2\dfrac{r^2}{1-r} \quad \Rightarrow \quad -\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k\right]\right| \ge -\dfrac{2r^2}{1-r}$

$$\Rightarrow \quad |(\mathbf{f} * \mathbf{g})(z_1) - (\mathbf{f} * \mathbf{g})(z_2)| \ge r - \rho - \left|\sum_{k=2}^{\infty} k a_k b_k \left[z_1^k - z_2^k\right]\right| \ge r - \rho - \frac{2r^2}{1-r} = r\left[1 - \frac{2r}{1-r}\right] - \rho$$

Observe that

$$r\left[1 - \frac{2r}{1-r}\right] - \rho > 0 \quad \Leftrightarrow \quad r\left[1 - \frac{2r}{1-r}\right] > \rho \ge 0$$

Consider

$$0 \le \rho < r\left[1 - \frac{2r}{1-r}\right] \quad \Rightarrow \quad 1 - \frac{2r}{1-r} > 0 \quad \Rightarrow \quad 1 > \frac{2r}{1-r}$$

$$\Rightarrow \quad 1 - r > 2r \quad (0 < r < 1 \quad \Rightarrow \quad -r > -1 \quad \Rightarrow \quad 1 - r > 1 - 1 = 0)$$

$$\Rightarrow \quad 1 > 3r \quad \Rightarrow \quad 3r < 1 \quad \Rightarrow \quad r < 3^{-1}$$

Thus we have $\quad 0 \le \rho = |z_1| \le |z_2| = r < 3^{-1}$

Thus $(\mathbf{f} * \mathbf{g})(z)$ is univalent in the open disc $\{z / |z| < 3^{-1} < 1\}$. //

Theorem 6 : Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is cap like function. Then $s_n(z, \mathbf{f}) = z + \sum_{k=2}^{n} a_k z^k$ are cap like function in the open disc $|z| < 0.25$.

Proof : $\mathbf{L}(z) = z(1-z)^{-1} = z\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k+1} = \sum_{k-1=0}^{\infty} z^{k-1+1} = \sum_{k=1}^{\infty} z^k = z + \sum_{k=2}^{\infty} a_k z^k$

where $a_k = 1 \ (k = 2, 3, 4, \cdots)$. Then by Theorem **4**,

$s_n(z, \mathbf{L}) = z + \sum_{k=2}^{n} z^k$ is cap like function in the open disc $|z| < 0.25$.

Replace z by $0.25z$ in $s_n(z, \mathbf{L})$, then $s_n(0.25z, \mathbf{L}) = 0.25z + \sum_{k=2}^{n} (0.25)^k z^k$ is also cap like function in the open disc $|0.25z| < 0.25$ *i.e.* $|z| < 1$.

By hypothesis, $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is cap like function in the open disc $|z| < 1$.

Convolution of $\mathbf{f}(z)$, $s_n(0.25z, \mathbf{L})$ is $\mathbf{f}(z) * s_n(0.25z, \mathbf{L}) = 0.25z + \sum_{k=2}^{n} a_k (0.25)^k z^k$.

By Theorem **4,** convolution of two cap like functions is cap like function.

Hence $\mathbf{f}(z) * s_n(0.25z, \mathbf{L})$ is cap like function in the open disc $|z| < 1$.

Replace z by $4z$ in $\mathbf{f}(z) * s_n(0.25z, \mathbf{L})$, then

$$\mathbf{f}(4z) * s_n(0.25 \times 4z, \mathbf{L}) = 0.25 \times 4z + \sum_{k=2}^{n} a_k (0.25)^k (4z)^k = z + \sum_{k=2}^{n} a_k z^k$$

is cap like function in the open disc $|4z| < 1$ *i.e.* $|z| < 0.25$ //