Matrix Transformation between Geometric Difference Sequence Spaces

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***Abstract*—**The aim of this paper is to determine matrix transformation between geometric difference sequence spaces.

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1. Introduction

In the area of non-Newtonian calculus pioneering work was carried out by Grossman and Katz [12] and is called as multiplicative calculus. The operations of multiplicative calculus are called as multiplicative derivative and multiplicative integral. We refer to Grossman and Katz [12], Stanley [17], Bashirov et al. [2, 3], Grossman [11, 13, 14] for different types of Non-Newtonian calculi and its applications. An extension of multiplicative calculus to functions of complex variables is handled by Bashirov and Riza [1], Uzer [20], Çakmak and Başar [8], Tekin and Başar [18], Türkmen and Başar [19]. Boruah and Hazarika [6, 7] discussed about basic properties of Bigeometric Differential Calculus and Bigeometric Integral Calculus.

Now a day geometric calculus is an alternative to the usual calculus of Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of addition. Almost all properties in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative calculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, mainly problems of price elasticity, resiliency, multiplicative growth etc. the use of multiplicative calculus is advocated instead of a traditional Newtonian calculus. To know better about Non-Newtonian calculus, we must have idea about different types of arithmetic and their generators.

Kizmaz [15] introduced the concept of difference sequence space over usual calculus. Et and Colak [16] studied generalized difference sequence spaces and discussed about matrix transformation between generalized difference sequence spaces. Boruah and Hazarika [4] introduced the geometric difference sequence spaces and studied about the geometric interpolation formulae using the geometric difference operator. In [5] Boruah and Hazarika studied generalized difference geometric sequence spaces and proved some interesting properties of these spaces.

2. α −Generator and Geometric Real Field

A $generator$ is a one-to-one function whose domain is $\mathbb{R}$(the set of real numbers) and range is a set $B \subset \mathbb{R}$. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic, and exponential function generates geometric arithmetic. As a generator, we choose the function α such that whose basic algebraic operations are defined as follows:

$$\begin{array}{lll}
\alpha-\text{addition} & x\dot{+}y & =\alpha[\alpha^{-1}(x)+\alpha^{-1}(y)] \\
\alpha-\text{subtraction} & x\dot{-}y & =\alpha[\alpha^{-1}(x)-\alpha^{-1}(y)] \\
\alpha-\text{multiplication} & x\dot{\times}y & =\alpha[\alpha^{-1}(x)\times\alpha^{-1}(y)] \\
\alpha-\text{division} & x\dot{/}y & =\alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] \\
\alpha-\text{order} & x\dot{<}y & \Leftrightarrow \alpha^{-1}(x)<\alpha^{-1}(y)
\end{array}$$

for $x, y \in A$, where A is a domain of the function α.

If we choose exp as a $\alpha - generator$ defined by $\alpha(z) = e^z$ for $z \in \mathbb{R}$ then $\alpha^{-1}(z) = \ln z$ and $\alpha - arithmetic$ turns out to geometric arithmetic.

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Geometric addition	$x \oplus y$	$= \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)]$	$= e^{(\ln x + \ln y)}$	$= x.y$
Geometric subtraction	$x \ominus y$	$= \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)]$	$= e^{(\ln x - \ln y)}$	$= x \div y$
Geometric multiplication	$x \odot y$	$= \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)]$	$= e^{(\ln x \times \ln y)}$	$= x^{\ln y}$
Geometric division	$x \oslash y$	$= \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)]$	$= e^{(\ln x \div \ln y)}$	$= x^{\frac{1}{\ln y}}$.

It is to be noted that if $x \ominus y$ exists if $y \neq 0$ and $x \oslash y$ exists if $y \neq 1$. Also, it is obvious that $\ln(x) < \ln(y)$ if $x < y$ for $x, y \in \mathbb{R}^+$. That is, $x < y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$. So, without loss of generality, we use $x < y$ instead of the geometric order $x \dot{<} y$.

Türkmen and Başar [19] defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows:

$$\begin{aligned} \mathbb{Z}(G) &= \{e^x : x \in \mathbb{Z}\} \\ \mathbb{R}(G) &= \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \setminus \{0\} \\ \mathbb{C}(G) &= \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}. \end{aligned}$$

Further $e^{-x} = \ominus e^x$ holds for all $x \in \mathbb{Z}^+$. Thus the set of all geometric integers turns out to the following:

$$\mathbb{Z}(G) = \{\ldots, e^{-3}, e^{-2}, e^{-1}, e^0, e^1, e^2, e^3, \ldots\} = \{\ldots, \ominus e^3, \ominus e^2, \ominus e, 1, e, e^2, e^3, \ldots\}.$$

If we take extended real number line, then $\mathbb{R}(G) = [0, \infty]$.

*Remark*2.0.1: $(\mathbb{R}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e, since

(1) $(\mathbb{R}(G), \oplus)$ is a geometric additive Abelian group with geometric zero 1,

(2) $(\mathbb{R}(G) \setminus 1, \odot)$ is a geometric multiplicative Abelian group with identity e,

(3) $\odot$ is distributive over $\oplus$.

But, $(\mathbb{C}(G), \oplus, \odot)$ is not a field, however, geometric binary operation $\odot$ is not associative in $\mathbb{C}(G)$. For, we take $x = e^{1/4}, y = e^4$ and $z = e^{(1+i\pi/2)} = ie$. Then

$$(x \odot y) \odot z = e \odot z = z = ie$$

but $$x \odot (y \odot z) = x \odot e^4 = e.$$

Let us define geometric positive real numbers and geometric negative real numbers as follows:

$$\begin{aligned} \mathbb{R}^+(G) &= \{x \in \mathbb{R}(G) : x > 1\} \\ \mathbb{R}^-(G) &= \{x \in \mathbb{R}(G) : x < 1\}. \end{aligned}$$

Then for all $x, y \in \mathbb{R}(G)$

- $x \oplus y = xy$
- $x \ominus y = x/y$
- $x \odot y = x^{\ln y} = y^{\ln x}$
- $x \oslash y$ or $\frac{x}{y}G = x^{\frac{1}{\ln y}}, y \neq 1$
- $x_1 \oplus x_2 \oplus \ldots \oplus x_n =_G \sum_{i=1}^{n} x_i = x_1.x_2 \ldots x_n$
- $x^{2_G} = x \odot x = x^{\ln x}$
- $x^{p_G} = x^{\ln^{p-1} x}$
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$
- $x^{-1_G} = e^{\frac{1}{\log x}}$
- $x \odot e = x$ and $x \oplus 1 = x$
- $e^n \odot x = x^n = x \oplus x \oplus \ldots$(n number of x)

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$$|x|^G = \begin{cases} x, & \text{if \$x > 1\$} \\ 1, & \text{if \$x = 1\$} \\ \frac{1}{x}, & \text{if \$0 < x < 1\$} \end{cases}$$

Thus $|x|^G \geq 1$.

- $\sqrt{x^{2_G}}^G = |x|^G$
- $|e^y|^G = e^{|y|}$
- $|x \odot y|^G = |x|^G \odot |y|^G$
- $|x \oplus y|^G \leq |x|^G \oplus |y|^G$
- $|x \oslash y|^G = |x|^G \oslash |y|^G$
- $|x \ominus y|^G \geq |x|^G \ominus |y|^G$
- $0_G \ominus 1_G \odot (x \ominus y) = y \ominus x$, *i.e.* in short $\ominus (x \ominus y) = y \ominus x$.

Definition 2.0.1. Let X be a non-empty set and $d^G: X \times X \to \mathbb{R}(G)$ be a function such that for all $x, y, z \in X$, the following axioms hold:

(NM1) $d^G(x,y) \geq 1$and $d^G(x,y) = 1$ iff $x = y$,
(NM2) $d^G(x,y) = d^G(y,x)$,
(NM3) $d^G(x,y) = d^G(x,z) \oplus d^G(z,y)$.

Then the pair (X, d^G) is called geometric metric space, d^G is called the geometric metric on X.

3. Geometric Infinite Matrices

A geometric infinite matrix $A = (a_{ij})$ of geometric real numbers is a sequence of real numbers defined by a function A from the set $\mathbb{N} \times \mathbb{N}$ into the geometric real field $\mathbb{R}(G)$, where $\mathbb{N}$ denotes the set of natural numbers. The geometric real number a_{ij} denotes the value of the function at $(i,j) \in \mathbb{N} \times \mathbb{N}$ and is called the entry of the matrix in the i^{th} row and j^{th} column.

Let $A = (a_{ij}) = (e^{\varepsilon_{ij}})$, and $B = (b_{ij}) = \left(e^{\delta_{ij}}\right)$ be two infinite geometric matrices where $\varepsilon_{ij}, \delta_{ij} \in \mathbb{R}$. We define the addition $\oplus$ and scalar multiplication $\odot$ of the infinite matrices as

$$\begin{aligned} A \oplus B &= (a_{ij} \oplus b_{ij}) = \left(e^{\varepsilon_{ij}} \oplus e^{\delta_{ij}}\right) = \left(e^{\varepsilon_{ij}+\delta_{ij}}\right) \\ \text{and } \lambda \odot A &= \left(\lambda \odot a_{ij}\right) = (e^{\mu} \odot e^{\varepsilon_{ij}}) = (e^{\mu.\varepsilon_{ij}}) \end{aligned}$$

where $\lambda = e^{\mu}$ is a geometric scalar in $\mathbb{R}(G)$. The product $A \odot B$ of the infinite matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined by

$$(A \odot B)_{ij} = \left({}_G\sum_{k=1}^{\infty} a_{ik} \odot b_{kj} \right) = \left({}_G\sum_{k=1}^{\infty} e^{\varepsilon_{ik}} \odot e^{\delta_{kj}} \right) = \left(e^{\left(\sum_{k=1}^{\infty} \varepsilon_{ik}\delta_{kj}\right)} \right) \qquad (3.1)$$

provided that the series on the right hand side of (3.1) converges for all $i, j \in \mathbb{N}$, where $(A \odot B)_{ij}$ denotes the entry of the matrix $A \odot B$ in the i^{th} row and j^{th} column. For simplicity in notation, summation without limit will represent the summation with limit from 1 to ∞. On the right hand side of (3.1) converges if and only if $\sum_{k=1}^{\infty} \varepsilon_{ik}\delta_{kj}$ is convergent for all $i, j, k \in \mathbb{N}$. However the series in (3.1) may not converge for some i, j, k; the product $A \odot B$ may not exist.

Definition 3.0.1. Consider the following system of an infinite number of equations in infinitely many unknowns $x_0, x_1, x_2, \ldots$ by ${}_G\sum_k a_{ik} \odot x_k = y_i$, for all $i \in \mathbb{N}$. If we construct a geometric infinite matrix $A = (a_{ij})$ with the coefficients a_{ij} of the unknowns x_k and denote the geometric-vector of unknowns by X and geometric-vector of constants by Y, then above sum can be expressed in matrix form as $A \odot X = Y$. Also $I^G \odot A = A \odot I^G = A$, where $I^G = (\delta_{ij})$ is called geometric-unit matrix and is defined by

$$\delta_{ij} = \begin{cases} e, & \text{if } \mathrm{i} = \mathrm{j} \\ 1, & \text{if } \mathrm{i} \neq \mathrm{j} \end{cases}$$

Example 3.0.1. Define the matrix $C_1^G = (c_{nk})$ by

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$$c_{nk} = \begin{cases} e^{\frac{1}{n}}, & \text{if} \quad 1 \leq k \leq n \\ 1, & \text{if} \quad k \geq n. \end{cases} \qquad (3.2)$$

That is

$$C_1^G = (c_{nk}) = \begin{pmatrix} e & 1 & \dots & \dots & \dots & \dots & \dots \\ \sqrt{e} & \sqrt{e} & 1 & \dots & \dots & \dots & \dots \\ \sqrt[3]{e} & \sqrt[3]{e} & \sqrt[3]{e} & 1 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sqrt[n]{e} & \sqrt[n]{e} & \sqrt[n]{e} & \dots & \sqrt[n]{e} & 1 & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \end{pmatrix}$$

The geometric-zero matrix θ^G is the matrix whose all entries are equal to 1. Thus from (3.2), it is obvious that $A \odot \theta^G = \theta^G = \theta^G \odot A$. Similar to the classical matrix multiplication $A \odot B = \theta^G$ does not imply that $A = \theta^G$ or $B = \theta^G$.

4. Geometric Matrix Transformations

Let $\mu_1, \mu_2 \subset \omega(G)$ be sequence spaces and $A = (a_{ij}) = (e^{\varepsilon_{ij}})$ be an infinite geometric-matrix. Then, we say that A defines a matrix mapping from μ_1 into μ_2 and denote it by writing $A: \mu_1 \to \mu_2$, if for every sequence $z = (z_k) = (e^{\eta_k}) \in \mu_1$ the sequence $A \odot z = (Az)_n$, the A −transformation of z, exists and is in μ_2, where

$$A \odot z = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & \dots \\ a_{21} & a_{22} & \dots & a_{2k} & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nk} & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix} \odot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\varepsilon 11} & e^{\varepsilon 12} & \dots & e^{\varepsilon 1k} & \dots & \dots \\ e^{\varepsilon 21} & e^{\varepsilon 22} & \dots & e^{\varepsilon 2k} & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ e^{\varepsilon n1} & e^{\varepsilon n2} & \dots & e^{\varepsilon nk} & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix} \odot \begin{pmatrix} e^{\eta 1} \\ e^{\eta 2} \\ \vdots \\ e^{\eta k} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} {}_G\sum_k e^{\varepsilon 1k} \odot e^{\eta k} \\ {}_G\sum_k e^{\varepsilon 2k} \odot e^{\eta k} \\ \vdots \\ {}_G\sum_k e^{\varepsilon nk} \odot e^{\eta k} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\left(\sum_k \varepsilon_{1k} . \eta_k\right)} \\ e^{\left(\sum_k \varepsilon_{2k} . \eta_k\right)} \\ \vdots \\ e^{\left(\sum_k \varepsilon_{nk} . \eta_k\right)} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} (Az)_1 \\ (Az)_2 \\ \vdots \\ (Az)_n \\ \vdots \end{pmatrix}$$

Thus, we transform the sequence $z = (z_k) = (e^{\eta_k})$ into the sequence $(Az)_n$ by

$$(Az)_n = \left({}_G\sum_k a_{nk} \odot z_k \right) = \left(e^{\left(\sum_k \varepsilon_{nk} . \eta_k\right)} \right) \qquad (4.1)$$

for $n \in \mathbb{N}$ and $\varepsilon_{ij}, \eta_i \in \mathbb{R}$. Thus $A \in (\mu_1 : \mu_2)$ if and only if the series ${}_G\sum_k a_{nk} \odot z_k$ of the right hand side of (4.1) geometrically converges $\forall n \in \mathbb{N}$ and $\forall z \in \mu_1$, and we have $A \odot z = \{(Az)_n\} \in \mu_2$ for all $z \in \mu_1$. On the other hand, we say $A \in (\mu_1 : \mu_2)$ if and only if the series $\sum_k \varepsilon_{nk} . \eta_k$ converges classically for all $k, n \in \mathbb{N}$. A sequence z is said to be A −summable to γ if $A \odot z$

geometrically converges to $\gamma \in \mathbb{R}(G)$, which we call $A-_G\lim$ of z. We denote n^{th} row of matrix $A = (a_{nk})$ by A_n for all $n \in \mathbb{N}$. Following Kadak and Efe [21], let $A = (a_{nk})$ be an infinite matrix of geometric real numbers, then

(a) Ordinary Summability: A sequence $z = (z_k) \in w(G)$ is said to be summable A to $\gamma \in \mathbb{R}(G)$ if the $A-_G\lim$ of z is $\gamma = e^u$ for all $u \in \mathbb{R}$, i.e. ${}_G\sum_n d^G((Az)_n, \gamma) = 1$ which implies that $\sum_k \varepsilon_{nk} . \eta_k \to u$ in classically for each $k, n \in \mathbb{N}$. The matrix A defines a summability method A, or a matrix transformation by (4.1).

(b) Absolute Summability: A sequence $z = (z_k) \in w(G)$ is said to be absolutely summable with index m to a number $\zeta \in \mathbb{R}(G)$ if the series ${}_G\sum_k a_{nk} \odot z_k$ in (4.1) geometrically converges for each $n \in \mathbb{N}$ and

$$ {}_G\sum_{n=1}^{\infty} d^G((Az)_n, \theta^G)^{m_G} = \zeta. (1 < m < \infty) $$

The Cesàro transform of a sequence $z = (z_k) \in w(G)$ is given by $C_1^G \odot z = \{(C_1^G z)_n\}_{n\in\mathbb{N}}]$. Now, we may state the Cesàro summability with respect to the geometric-calculus, which is analogous to the classical Cesàro summability.

Example 4.0.1 [19] Let $z = (z_k)$ be an infinite sequence of geometric numbers defined by

$$ z_k = \begin{cases} e, & \text{if } k \text{ is even,} \\ e^{-1}, & \text{if } k \text{ is odd.} \end{cases} $$

It is obvious that $z = (z_k) \in l_\infty^G \setminus c^G$. Then, since $1 \leq \|(C_1^G z)_n\|^G \leq e^{\frac{1}{n}}$ for all $n \in \mathbb{N}, \lim_{n\to\infty}(C_1^G z)_n = \theta^G$. It means that the geometric divergent sequence (z_k) is C_1^G $-$summable to θ^G.

Türkmen and Başar [19] have introduced the sets $\omega^G, l_\infty^G, c^G, c_0^G$ and l_p^G for all, bounded, convergent, null and absolutely p $-$summable sequences over the geometric field $\mathbb{C}(G)$ as mentioned above. Then l_∞^G, c^G, c_0^G and l_p^G are subspaces of ω^G.

Theorem 4.0.1 The following statements hold:

(a) l_∞^G, c^G, c_0^G and l_p^G are sequence spaces.

(b) If λ denotes any of the spaces $l_\infty(G), c(G), c_0(G)$ and l_p^G and $z = (z_k), t = (t_k) \in \lambda$, then define d_∞^G on λ by $d_\infty^G = \sup_{k\in\mathbb{N}}|z_k \ominus t_k|^G$. Then (λ, d_∞^G) is a complete metric space.

(c) The spaces $l_\infty(G), c(G), c_0(G)$ and l_p^G are Banach spaces with the norm $\|z\|^G$ defined by

$$ \|z\|^G = \sup_{k\in\mathbb{N}}|z_k|^G; z = (z_1, z_2, \ldots, z_k, \ldots) \in \lambda. $$

(d) The space l_p^G is a Banach space with respect to the norm $\|z\|_p^G$ defined by

$$ \|z\|^G = \left[{}_G\sum_{k=1}^{\infty} \left(|z_k|^G\right)^{p_G} \right]^{\frac{e}{p}G} ; z = (z_1, z_2, \ldots, z_k, \ldots) \in l_p^G $$

Note: It is to be noted that $x^{p_G} = x^{ln^{p-1}x}$ and $x^{\frac{e}{p}G} = \sqrt{px}^G = e^{(\ln x)^{\frac{1}{p}}}$ is the geometric p^{th} root of x.

Theorem 4.0.2. The spaces $l_\infty^G(\Delta_G), c^G(\Delta_G)$ and $c_0^G(\Delta_G)$ are Banach spaces with respect to the norm

$$ \|x\|_{\Delta_G}^G = |x_1|^G \oplus \|\Delta_G x\|_\infty^G. $$

Theorem 4.0.3. Let μ denotes one of the spaces $l_\infty^G(\Delta_G), c^G(\Delta_G)$ or $c_0^G(\Delta_G)$ and $z = (z_k) = (e^{u_k}), t = (t_k) = (e^{v_k}) \in \mu$.Define d_∞^G on the space μ by

$$ \begin{aligned} d_\infty^G : \mu \times \mu & \to \mathbb{R}(G) \\ (z, t) & \to d_\infty^G(z, t) = \sup_{n\in\mathbb{N}} \left\{ d^G\left({}_G\sum_{k=1}^{\infty} z_k, {}_G\sum_{k=1}^{\infty} t_k \right) \right\} \end{aligned} $$

Where d^G is the metric defined in **Definition 2.0.1**. Then (μ, d_∞^G) is a complete metric space.

Proof: Proof is obvious.

Now we give some characterization of some matrix classes and state the necessary and sufficient conditions on geometric matrix transformations using the results of Köthe-Toeplitz duals .

Theorem 4.0.4. The following statements hold: (i) $A = (a_{nk}) \in (l_\infty^G : l_\infty^G)$ if and only if

$$ M = \sup_{n\in\mathbb{N}} {}_G\sum_k |a_{nk}|^G < \infty \qquad (4.2) $$

(ii) $A=(a_{nk})\in(c^G:l_\infty^G)$ if and only if (4.2) holds.
(iii) $A=(a_{nk})\in(c_0^G:l_\infty^G)$ if and only if (4.2) holds.
(iv) $A=(a_{nk})\in(l_p^G:l_\infty^G)$ if and only if

$$C=\sup_{n\in\mathbb{N}}\ {}_G\sum_k\left[|a_{nk}|^G\right]^{p_G}<\infty \qquad (4.3)$$

Proof: (i) Suppose that the condition (4.2) holds and $x=(x_k)\in l_\infty^G$. Since $(a_{nk})_{k\in\mathbb{N}}\in\{l_\infty^G\}^\beta=l_1^G$ for every fixed $n\in\mathbb{N}$, the geometric A −transformation of x exits. Then

$$\sup_{n\in\mathbb{N}}d^G((Ax)_n,\theta^G)=\sup_{n\in\mathbb{N}}d^G\left({}_G\sum_k a_{nk}\odot x_k,\theta^G\right)\leq\|x\|_\infty^G\odot\ {}_G\sum_k|a_{nk}|^G<\infty,$$

This implies that $A\odot x\in l_\infty^G$.

Conversely, let $A=(a_{nk})\in(c^G:l_\infty^G)$. Put $A\odot x=\{(Ax)_n\}_{n\in\mathbb{N}}$ and observe that $((Ax)_n)$ is a sequence of bounded linear operators on l_1^G such that $\sup_n d^G((Ax)_n,\theta^G)<\infty$. Hence the results are obtained similarly from an application of Banach-Steinhaus theorem in classical.

Similarly, we can prove (ii), (iii) and (iv).

Example.4.0.2 Let $(x_k)=(e^{\varepsilon_k})\in l_\infty^G$ and matrix $A=(a_{nk})$ is defined as

$$a_{nk}=\begin{cases}x_k, & \text{if } k=n,\\ 1, & \text{if } k\neq n,\end{cases}$$

for all $k,n\in\mathbb{N}$. Then $|a_{nn}|^G=|e^{\varepsilon_n}|^G=e^{|\varepsilon_n|}$ and $|a_{nk}|^G=1$ for $k\neq n$. If we consider $(x_k)\in l_\infty^G$, we obtain

$$\sup_{n\in\mathbb{N}}\ {}_G\sum_k|a_{nk}|^G=\sup_{n\in\mathbb{N}}\{e^{|\varepsilon_1|},e^{|\varepsilon_2|},e^{|\varepsilon_3|},\ldots,e^{|\varepsilon_n|},\ldots\}<\infty$$

for all $\varepsilon_K\in\mathbb{R}$. Then from Theorem Theorem4.0.4(i), $A=(a_{nk})\in(l_\infty^G:l_\infty^G)$.

In (4.1), the matrix transformation

$$(Az)_n=\left({}_G\sum_k a_{nk}\odot z_k\right)=\left(e^{\left(\sum_k\varepsilon_{nk}\cdot\eta_k\right)}\right)$$

suggests two problems:

(i) to determine the family K of matrices (which are called Kojima-matrices) such that convergence is not destroyed by the corresponding transformations;

(ii) to determine that subclass T (which are called Toeplitz-matrices) of K for which the value of the limit of any convergent sequence is invariant.

The first problems was solved by Kojima and Schur and the second problem was solved by Silverman and Toeplitz in terms of classical sequence space. We state and prove the Kojima-Schur theorem (**Theorem 4.0.5**) and Silverman-Toeplitz theorem (**Corollary 4.0.8**) with respect to the geometric calculus which gives the necessary and sufficient conditions for an infinite matrix that maps c^G into itself. A matrix satisfying the the conditions of Kojima-Schur is called a conservative matrix or convergence preserving matrix.

Theorem 4.0.5 (Kojima-Schur) $A=(a_{nk})\in(c^G:c^G)$ *if and only if*
$(i)\ M=\sup_{n\in\mathbb{N}}\ {}_G\sum_k|a_{nk}|^G<\infty)$
and there exits $\alpha_k,l\in\mathbb{R}(G)$ *such that*

$$(ii)\quad {}_G\sum_k a_{nk}=\alpha_k\ for\ fixed\ k\in\mathbb{N},$$

$(iii)\quad {}_G\sum_{n\to\infty}\ {}_G\sum_k\quad a_{nk}=l.$

Proof. Suppose that the conditions (i), (ii) and (iii) hold and $x=(x_k)\in c^G$ with $x_k\xrightarrow{G}s\in\mathbb{R}(G)$as $k\to\infty$. Then since $(a_{nk})_{k\in\mathbb{N}}\in\{c^G\}^\beta=l_1^G$ for each $n\in\mathbb{N}$ the geometric A −transform of x exits. In this situation, the equality

$$ {}_G\sum_k a_{nk}\odot x_k=\left\{{}_G\sum_k a_{nk}\odot(x_k\ominus s)\right\}\oplus\left\{s\odot\ {}_G\sum_k a_{nk}\right\}\qquad (4.5)$$

holds for each $n\in\mathbb{N}$. In (4.5), since the terms of the right hand side tends to ${}_G\sum_k a_{nk}\odot(x_k\ominus s)$ by condition (ii) and second term on the right hand side tends to $l\odot s$ by (iii) as $n\to\infty$, in the sense of geometric limit, we have

$$_G\sum_{n\to\infty} \ _G\sum_k a_{nk}\odot x_k = \left\{ \ _G\sum_k \alpha_k\odot(x_k\ominus s)\right\}\oplus l\odot s$$

Hence, $Ax\in c^G$. That is, the conditions are sufficient. Conversely, suppose that $A=(a_{nk})\in(c^G:c^G)$. Then $A\odot x$ exists for every $x\in c^G$. Let $e^G=(e,e,e,\ldots\ldots)$ and $e_n^G=(e_k^{(n)})$ where

$$e_k^{(n)}=\begin{cases} e, & \text{if } k=n,\\ 1, & \text{if } k\neq n\end{cases}$$

The necessary conditions (ii) and (iii) is immediate by taking $x=e_k^G$ and $x=e^G$, respectively. Since $c^G\subset l_\infty^G$, the necessity of condition (i) is obtained from the first condition of the Theorem 4.0.4.

Theorem 4.0.6. $A=(a_{nk})\in(c_0^G:c^G)$ if and only if

$$(i) M=\sup_{n\in\mathbb{N}} \ _G\sum_k |a_{nk}|^G<\infty,$$

$$(ii)_G\sum_{n\to\infty} d^G(a_{nk},\alpha_k)=1 for (\alpha_k)\in\omega(G)$$

for each $k\in\mathbb{N}$. *If* $A=(a_{nk})\in(c_0^G:c^G)$, *then* $(\alpha_k)\in l_1^G$ *and*

$$_G\sum_{n\to\infty} \ _G\sum_k a_{nk}\odot z_k = \ _G\sum\alpha_k\odot z_k.$$

Proof. Suppose (i) and (ii) hold. Then there exists $n_t\in\mathbb{N}$ for $t\in\mathbb{N}$ and $\varepsilon>1$ such that

$$_G\sum_{k=1}^{t} d^G(a_{nk},\alpha_k)<\varepsilon \qquad \text{for all } n\geq n_t.$$

Since

$$_G\sum_{k=1}^{t} d^G(\alpha_k,1)\underset{G}{\leq}\sum_{k=1}^{t} d^G(a_{nk},\alpha_k)\underset{G}{\oplus}\sum_{k=1}^{t} d^G(a_{nk},1)\leq\varepsilon\oplus M$$

for $n\geq n_t$ by (ii) it can be shown that $(\alpha_k)\in l_1^G$ and $\ _G\sum_{k=1}^{t} d^G(\alpha_k,1)\leq M_0$. Let $z=(z_k)\in c_0^G$. Then, we can choose $k_0\in\mathbb{N}$ for $\varepsilon_1>1$ such that $d^G(z_k,1)<\varepsilon_1$ for each fixed $k\geq k_0$. Also, since $a_{nk}\xrightarrow{G}\alpha_k$ as $n\to\infty$ by (ii), we have $a_{nk}\odot z_k\xrightarrow{G}\alpha_k\odot z_k$ as $n\to\infty$ for each fixed $k\in\mathbb{N}$. That is to say that $\ _G\sum_{n\to\infty} \ d^G(a_{nk}\odot z_k,a_k\odot z_k)=1$. Hence there exists an $N=N(k_0)\in\mathbb{N}$ such that $\ _G\sum_{k=1}^{k_0} d^G(a_{nk}\odot z_k,\alpha_k\odot z_k)<\varepsilon_2$ for all $n\geq N$. Thus, since

$$\begin{aligned}
& d^G\left(\ _G\sum_k \ a_{nk}\odot z_k, \ _G\sum_K \ \alpha_k\odot z_k\right)\\
& \leq \ _G\sum_k \ d^G(a_{nk}\odot z_k,\alpha_k\odot z_k)\\
= & \ _G\sum_{k=1}^{k_0} d^G(a_{nk}\odot z_k,\alpha_k\odot z_k)\oplus \ _G\sum_{k=k_0+1}^{\infty} d^G(a_{nk}\odot z_k,\alpha_k\odot z_k)\\
& \leq\varepsilon_2\oplus \ _G\sum_{k=k_0+1}^{\infty}[d^G(a_{nk}\odot z_k,1)\oplus d^G(\alpha_k\odot z_k,1)]\\
& =\varepsilon_2\oplus \ _G\sum_{k=k_0+1}^{\infty} d^G(a_{nk},1)\odot d^G(z_k,1)\oplus \ _G\sum_{k=k_0+1}^{\infty} d^G(\alpha_k,1)\odot d^G(z_k,1)\\
& =\varepsilon_2\oplus\left\{\varepsilon_1\odot\left(\ _G\sum_{k=k_0+1}^{\infty} d^G(a_{nk},1)\oplus \ _G\sum_{k=k_0+1}^{\infty} d^G(\alpha_k,1)\right)\right\}\\
& =\varepsilon_2\oplus\{\varepsilon_1\odot(M\oplus M_0)\}
\end{aligned}$$

for all $n\geq N$, the series $\ _G\sum_k a_{nk}\odot z_k$ are geometrically convergent for each $n\in\mathbb{N}$ and $_G\sum_k a_{nk}\odot z_k \xrightarrow{G} _G\sum_k \alpha_k\odot z_k,$ as $n\to\infty$. This implies that $A\odot z\in c^G$.

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Conversely, let $A = (a_{nk}) \in (c_0^G : c^G)$ and $z = (z_k) \in c_0^G$. Then, since $A \odot z \in c^G$ exists and the inclusion $(c_0^G : c^G) \subset (c_0^G : l_\infty^G)$ holds, the necessity of (i) is trivial by (iii) of Theorem [thm1]. Now, if we take the sequence $e_n^G = \{e^{(n)_k}\} \in c_0^G$ then $A \odot e_n^G = \{a_{nk}\}_{n=1}^{\infty} \in c^G$ holds for each fixed $k \in \mathbb{N}$, i.e. the condition (ii) is also necessary. Thus, the proof is complete.
As a consequence of Theorem 4.0.6, we have

Corollary 4.0.7. $A = (a_{nk}) \in (c_0^G : c_0^G)$ *if and only if*

$$(i) M = \sup_{n\in\mathbb{N}} {}_G\sum_k |a_{nk}|^G < \infty,$$

$$(ii) {}_G\sum_{n\to\infty} d^G(a_{nk}, \alpha_k) = 1 \; with \; \alpha_k = 1 \; for \; all \; k \in \mathbb{N}$$

Example 4.0.3. Let $k, n, r \in \mathbb{N}$ and $r \geq 0$. The Cesaro means of order r is defined by the matrix $C_r^G = (c_{nk}^{(r)})$ where

$$c_{nk}^{(r)} = \begin{cases} e^{\binom{n-k+r-1}{n-k}/\binom{n+r}{n}}; & \text{if } 0 \leq k \leq n \\ 1 & \text{if } k > n. \end{cases}$$
$$= \begin{cases} e^{\frac{n!(n-k+r-1)!r}{(n-k)!(n+r)!}} & \text{if } 0 \leq k \leq n \\ 1 & \text{if } k > n. \end{cases}$$

Taking $r = 2$ we have

$$c_{nk}^{(2)} = \begin{cases} e^{\frac{2(n-k+1)}{(n+1)(n+2)}} & \text{if } 0 \leq k \leq n \\ 1 & \text{if } k > n. \end{cases}$$

Then we obtain an infinite matrix as follows:

$$C_2^G = \begin{pmatrix} e & 1 & 1 & 1 & 1 & \cdots \\ e^{2/3} & e^{1/3} & 1 & 1 & 1 & \cdots \\ e^{3/6} & e^{2/6} & e^{1/6} & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{\frac{2}{n+2}} & e^{\frac{2n}{(n+1)(n+2)}} & \cdots & e^{\frac{2(n-k+1)}{(n+1)(n+2)}} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It can be concluded that $\sup_{n\in\mathbb{N}} {}_G\sum_k \left|c_{nk}^{(2)}\right|^G < \infty$ for all $k \in \mathbb{N}$ and (i) holds. Also

$${}_G\sum_{n\to\infty} \left|c_{nk}^{(2)}\right|^G = {}_G\sum_{n\to\infty} \left|e^{\frac{2(n-k+1)}{(n+1)(n+2)}}\right|^G = 1.$$

So (ii) also holds with $\alpha_k = 1$ for all $k \in \mathbb{N}$. Therefore $C_2^G \in (c_0^G : c_0^G)$.

A Toeplitz matrix that satisfies Silverman-Toeplitz theorem is also called regular matrix. The class of geometric Toeplitz matrices will be denoted by $(c^G : c^G; p)$.

Corollary 4.0.8 (Silverman-Toeplitz). $A = (a_{nk}) \in (c^G : c^G; p)$ *if and only if*

$$(i) M = \sup_{n\in\mathbb{N}} {}_G\sum_k |a_{nk}|^G < \infty,$$

$$(ii) {}_G\sum_{n\to\infty} d^G(a_{nk}, \alpha_k) = 1 \; for \; fixed k \in \mathbb{N},$$

$$(iii) \; {}_G\sum_{n\to\infty} {}_G\sum_k a_{nk} = 1.$$

with $\alpha_k = 1$ *for all* $k \in \mathbb{N}$ *and* $l = e$, *respectively.*

Remark: Example 4.0.1 can be considered as an example of Silverman-Toeplitz theorem. Because the conditions (i) and (ii) holds with $\alpha_k = 1$. Also, conditon (iii) holds as

$$
\begin{aligned}
{}_G\sum_{n\to\infty} {}_G\sum_k \left|c_{nk}^{(2)}\right|^G &= {}_G\sum_{n\to\infty} {}_G\sum_k \left|e^{\frac{2(n-k+1)}{(n+1)(n+2)}}\right|^G \\
&= {}_G\sum_{n\to\infty} {}_G\sum_k e^{\left|\frac{2(n-k+1)}{(n+1)(n+2)}\right|} \\
&= {}_G\sum_{n\to\infty} e^{\sum_k \left|\frac{2(n-k+1)}{(n+1)(n+2)}\right|} \\
&= e.
\end{aligned}
$$

Theorem 4.0.9. $A = (a_{nk}) \in (l_\infty^G : c_0^G)$ *if and only if*

$$
{}_G\sum_{n\to\infty} {}_G\sum_k d^G(a_{nk}, 1) = 1. \qquad (4.6)
$$

Proof. Let $A = (a_{nk}) \in (l_\infty^G : c_0^G)$ and $u = (u_k) \in l_\infty^G$. Then, the series ${}_G\sum_k a_{nk} \odot u_k$ geometrically converges to 1 for each fixed $n \in \mathbb{N}$, since $A \odot u$ exists. Hence, $A_n = \{a_{nk}\}_{k=0}^\infty \in \{l_\infty^G\}^\beta$ for all $n \in \mathbb{N}$. Define the sequence $u = (u_k) \in l_\infty^G$ by $u_k = (e, e, e, e, \ldots\ldots)$ for all $k \in \mathbb{N}$. Then, $A \odot u \in c_0^G$ which yields for all $n \in \mathbb{N}$ that

$$
{}_G\sum_{n\to\infty} {}_G\sum_k a_{nk} \odot u_k = {}_G\sum_{n\to\infty} {}_G\sum_k a_{nk} \odot e \underset{G}{=} \sum_{n\to\infty} {}_G\sum_k a_{nk} = 1
$$

Also, we have

$$
{}_G\sum_{n\to\infty} {}_G\sum_k d^G(a_{nk}, 1) = {}_G\sum_{n\to\infty} {}_G\sum_k |a_{nk}|^G = {}_G\sum_k {}_G\sum_{n\to\infty} |a_{nk}|^G = 1.
$$

Conversely, suppose that (4.6) holds and $u = (u_k) \in l_\infty^G$. Then, since $A_n = \{a_{nk}\}_{k=0}^\infty \in \{l_\infty^G\}^\beta = l_1^G$ for each $n \in \mathbb{N}$, $A \odot u$ exists. Therefore, using (4.6), it can be observed that

$$
\begin{aligned}
{}_G\sum_{n\to\infty} {}_G\sum_k d^G((Au)_n, 1) &= {}_G\sum_{n\to\infty} d^G\left({}_G\sum_k a_{nk} \odot u_k, 1\right) \\
&= {}_G\sum_{n\to\infty} {}_G\sum_k d^G(a_{nk} \odot u_k, 1) \\
&= {}_G\sum_{n\to\infty} {}_G\sum_k d^G(a_{nk}, 1) \odot d^G(u_k, 1) \\
&\le \sup_{k\in\mathbb{N}} d^G(a_{nk}, 1) \odot {}_G\sum_{n\to\infty} {}_G\sum_k d^G(u_k, 1) = 1.
\end{aligned}
$$

This implies that $A \odot u \in c_0^G$.

5. Conclusion

Though Tekin, Basar, Ugur developed generalized non-Newtonian complex sequence spaces, it is also essential to develop geometric sequence spaces separately. Because, in some cases properties considered in generalized complex field may not be true for special cases. For example, if A and B are two infinite matrices, then the classical sum $A + B$ always exists, but the geometric sum $A \oplus B$ may not exist.

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