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Lagrangian Invariance in Gauge Theories

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Abstract—The pedagogical introduction to the global and local symmetries, invariance of Lagrangian and therefore, of the physical laws which predict the existence of a covariant derivative,is presented. QED, QCD and the electroweak theories are summarized as the examples of gauge theories.

Keywords- *Gauge Theory, SU(3), Abelian Symmetry, QED, Covariance.*

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I. INTRODUCTION

H Weyl in 1919 [1]introduced the concept of Gauge theory and the principle of local gauge invariance for the first time in an attempt to model the electromagnetic and the gravitational field as geometrical properties of space-time. A gauge theory is a theory where the action is invariant under a continuous group symmetry that depends on space-time. When the symmetry group depends on space-time, it is called a local symmetry. The transformation that depends on space-time is called a gauge transformation [2, 3]. Gauge theories are used to describe and unify the three of the four fundamental forces of nature, namely the electromagnetism, the weak force and the strong force.

This brief discussionaims to complement the available study material on Gauge Theoriesand provide a short insight into what gauge theories are, how they work and why we use them.Historically, while trying to explain the quantum effects of electrodynamics, it was found QED can be explained by a U(1) abelian gauge theory. Yang and Mills [4] then generalised this abelianU(1) gauge theory to the non-abelian gauge theory case. It is shown that gauge transformations leave the action and the classical equations of motion invariant. The phase of the wave-function itself is unobservable in quantum mechanics. Only the phase differences are observable e.g. via interference phenomena. The phase difference is visible in the Aharonov–Bohm effect [5] whose scheme is depicted in Fig.1.

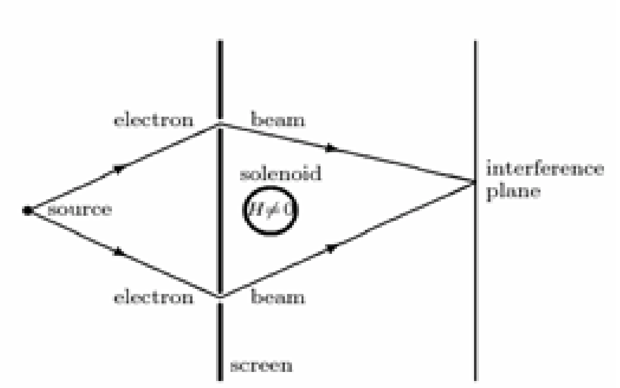

Figure 1. Observing interference effects of the Aharonv-Bohm experiment

II. EXAMPLES OF GAUGE THEORIES

A. *Quantum Electrodynamics (QED)*

QED has the gauge group $U(1)_{EM}$. The number of gauge fields is $\dim(U(1)_{EM}) = 1$. This gauge field is the photon.It couples to charged leptons and quarks. Does SSB occur: No. So the photon remains massless.

B. *Quantum Chromodynamics (QCD)*

QCD has the gauge group $\mathbf{SU(3)_{colour}}$. A gauge transformation is $U \in SU(3)_{colour}$. QCD offers a new way of thinkingabout matter. Every quark field of flavour f, say $\Psi^f(x)$, has an associated color of red, green or blue. Define

$$\chi^f_{red}(x) = \begin{pmatrix} \Psi^f_{red}(x) \\ \Psi^f_{green}(x) \\ \Psi^f_{blue}(x) \end{pmatrix}$$

The gauge invariant Lagrangian can be constructed, of the form

$$\mathcal{L} = \bar{\chi}^f\left(i\gamma^\mu D_\mu - m\right)\chi^f$$

This Lagrangian is invariant under $SU(3)_{colour}$, i.e. $\chi^f(x) \to U(x)\chi^f(x)$where $U(x) \in SU(3)_{colour}$. The number of gaugefields is equal to the dimensions of $SU(3)_{colour}$ (or $\dim(SU(3)_{colour})$ which is 8. These are the eight gluon fields that couple to the quarks, holding them together to formnon-pertubative bound states called hadrons. Does SSB occur? No, so the gluons remain massless.

C. *ElectroWeak (EW) Theory*

The EW theory has the gauge group $SU(2)_{left}\times U(1)_{hypercharge}$. The number of gauge fields is equal to $\dim(SU(2)_{left}\times U(1)_{hypercharge}) = 4$. These gauge fields are the W^a_μ, B_μ, a = 1,2,3.Does SSB occur? Yes. The gauge group of Electroweak theory i.e $SU(2)_{left}\times U(1)_{hypercharge}$interacts with a complex doublet. A complex doubletthas four real scalar field components. The symmetry group is then spontaneously broken to $U(1)_{QED}$. Each broken generatorproduces a goldstone boson. The number of goldstone bosons is, therefore, equal to the number of broken generators or lost dimensions i.e. dim(G/H) =

$\dim(SU(2)_{left} \times U(1)_{hypercharge}) - \dim(U(1)_{QED}) = 4 - 1 = 3$. These three goldstone bosons are "eaten" by three gauge fields to become massive (in unitarygauge). These massive gauge fields are called $W_\mu^\pm, Z_\mu^0$. The $W_\mu^\pm$ couple to left handed matter causing flavour changing processes like beta decay, Z_μ^0couples to all matter particles.The massless gauge field is called the photon which photon couples to charged matter only. We still have one massive real scalar field left after SSB, which we call the Higgs.

III. DERIVATION OF GAUGE THEORIES

A symmetry is a transformation of the fields that leaves the action (and hence "physics") invariant. A global symmetry is a symmetry that does not depend on space-time. Global symmetries give rise to conserved currents and charges as described by Noether's theorem. On the other hand, gauge symmetry is a continuous local symmetry where the symmetry group is continuous and depends on space-time. Gauge symmetries introduce gauge fields to the theory which mediate a force.

The following discussion on global symmetries, before we discuss the gauge symmetries, helpsto clarify the concepts which lie behind the answers to the questions like what exactly is a gauge theory and how gauge fields appear.

A. *Global symmetries*

An example of a global symmetry is the phase transformation is

$$\Psi(x) \rightarrow e^{i\alpha}\Psi(x); \bar{\Psi}(x) \rightarrow e^{-i\alpha}\bar{\Psi}(x) \qquad (1)$$

where $\alpha$is a parameter which does not depend on spacetime, x.We know, a free fermion field is described by the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \qquad (2)$$

Using transformation laws for $\Psi(x)$ and $\bar{\Psi}(x)$ as given in Eq.(1). Lagrangian $\mathcal{L}$ will transform to

$$\mathcal{L} \rightarrow \bar{\Psi}(x)e^{-i\alpha}(i\gamma^\mu \partial_\mu - m)e^{i\alpha}\Psi(x) = \mathcal{L} \qquad (3)$$

Since αdoes not depend on space-time, x, we can commute it past the derivative and so Lagrangian $\mathcal{L}$remains invariant.

Let $\Psi_1(x)$and $\Psi_2(x)$be two fields both with same mass m1 = m2 = m. Let us define the complete wave function for the combined field

$$\chi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \text{ and } \bar{\chi}(x) = (\bar{\Psi}_1(x), \bar{\Psi}_2(x)) \qquad (4)$$

Then the new Lagrangian for the two free fields is

$$\mathcal{L} = \overline{\Psi_1}(x)(i\gamma^\mu \partial_\mu - m)\Psi_1(x) + \overline{\Psi_2}(x)(i\gamma^\mu \partial_\mu - m)\Psi_2(x)$$
$$= (\bar{\Psi}_1(x), \bar{\Psi}_2(x))(i\gamma^\mu \partial_\mu - m)\Psi_1(x)\begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$$
$$= \bar{\chi}(x)[(i\gamma^\mu \partial_\mu - m)I_{2\times2}]\chi(x) \qquad (5)$$

where γ^μ is a 4 × 4 matrix and a scalar with respect to the spinor doublet $\chi(x)$.

$\chi(x)$ is atwo component spinor-column vector, where each component is a 4-component column vector Ψ. Therefore $\chi(x)$ is an 8 component column vector.Further $(i\gamma^\mu \partial_\mu - m)$is a scalar and $(i\gamma^\mu \partial_\mu - m)I_{2\times2}$ is a scalar multiplied by the identity matrix. The I_{2x2} can be left out but we shall leave it in, to be explicit.

Transformation inU(2) global symmetry is given by

$$\chi(x) \rightarrow \Omega\chi(x) \quad \text{where} \quad \Omega \in U(2)$$

It is obvious that Ω is a 2x2 matrix or can be a tensor representation.

$$\bar{\chi}(x) \rightarrow \bar{\chi}(x)\Omega^\dagger \quad \text{and so}$$
$$\mathcal{L} \rightarrow \bar{\chi}(x)\Omega^\dagger[(i\gamma^\mu \partial_\mu - m)I_{2\times2}]\Omega\chi(x) = \mathcal{L} \qquad (6)$$

As discussed above $\chi(x)$ is a two component spinor-column vector and $\gamma^\mu$is a scalar withrespect to χ. Therefore,γ^μcan commute pastΩ..AsΩalso doesnot depend on space-time, it also commute past the derivative. Therefore $\mathcal{L}$is invariant.

This concept of U(2) symmetry can be extended to a U(N) symmetry straightforwardly. The N fermions fields with the samemassisput into an N component column vector $\chi(x)$ and just redoing everything we have done.The transformation of SU(3) is an isospin symmetry, considering the up (u), down (d) and strange(s) quarks as having equal mass,and is one of the examples.

It will however be interesting to see what would happen to our symmetryif thetransformation does depend on space-time? Then,Ω will not commute past the derivative.

B. *Gauge symmetry*

A local symmetry is a symmetry that depends on space-time, x. A gauge symmetry is a local symmetry where thesymmetry group is continuous e.g.U(N), SU(N).

Let's say we have an N component column vector of fermion fields with identical masses,

$$\chi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix} \qquad (7)$$

Then, just as in the previous example, the Lagrangian

$$\mathcal{L} = \bar{\chi}(x)[(i\gamma^\mu \partial_\mu - m)I_{N\times N}]\chi(x) \qquad (8)$$

will be invariant under a global U(N) transformation given by

$$\chi(x) \rightarrow \Omega\chi(x) \quad \text{where} \quad \Omega \in U(N)$$
$$\bar{\chi}(x) \rightarrow \bar{\chi}(x)\Omega^\dagger \qquad (9)$$

The transformation Ωpreserves local symmetry, that means it will be a function of space and time or $\Omega \rightarrow \Omega(x)$. How does the Lagrangian transform now?

$$\mathcal{L} = \bar{\chi}(x)\Omega^\dagger(x)[i\gamma^\mu \partial_\mu - m]\Omega(x)\chi(x)$$
$$\mathcal{L} = \bar{\chi}(x)\Omega^\dagger(x)[i\gamma^\mu \partial_\mu(\Omega(x)\chi(x)) - m\Omega(x)\chi(x)]$$
$$\mathcal{L} = \bar{\chi}(x)\Omega^\dagger(x)[i\gamma^\mu\Omega(x)\partial_\mu\chi(x) + i\gamma^\mu \partial_\mu(\Omega(x))\chi(x) - m\Omega(x)\chi(x)]$$
$$= \mathcal{L} + \bar{\chi}(x)\Omega^\dagger(x)i\gamma^\mu \partial_\mu(\Omega(x))\chi(x) \qquad (10)$$

which is not invariant because we cannot commute the transformation matrix $\Omega(x)$past the derivative because it is a space-time function.

This seems like a dead end. The local symmetry isnot even symmetry! But let's say we are really adamant that "physics" is invariant under the transformation

$$\chi(x) \rightarrow \Omega(x)\chi(x) \qquad (11)$$

The problem is that derivative∂_μdoes not transform covariantly.

$$\partial_\mu(\chi(x)) \nrightarrow \Omega(x)\partial_\mu\chi(x)$$

However, we have to construct a covariant derivative, say D_μ which will transform covariantly in order to make our Lagrangian invariant under the transformations. Therefore

$$D_\mu(\chi(x)) \to \Omega(x) D_\mu \chi(x) \qquad (12)$$

Let us consider the ordinary derivative in some direction n^μ:

$$\partial_\mu \chi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\chi(x+\epsilon) - \chi(x)\right) \qquad (13)$$

where ϵ is a small displacement in n^μ direction.

The problem arises because $\chi(x+\epsilon)$ and $\chi(x)$ do not transform identically because of spacetime dependence of the transformation $\Omega(x)$.

Therefore transformation equation for $\chi(x+\epsilon)$ is

$$\chi(x+\epsilon) \to \Omega(x+\epsilon)\chi(x+\epsilon)$$

and not $\chi(x+\epsilon) \to \Omega(x)\chi(x+\epsilon)$

whereas

$$\chi(x) \to \Omega(x)\chi(x)$$

Therefore, the derivative ∂_μ, as defined in Eq.(13) loses its normal meaning. The two fields at different spacetime points cannot be compared because of change in value of the co-ordinates ‘x’ themselves. We need to construct a new derivative with the two field values $\chi(x)$ and $\chi(x+\epsilon)$ transforming in the same way. Let us define transformation $U(y, x) \in SU(N)$, called parallel transport, which itself transforms under a gauge transformation as

$$U(y,x) \to \Omega(y) U(y,x) \Omega^\dagger(x) \qquad (14)$$

Therefore

$$U(y,x)\chi(x) \to \Omega(y)U(y,x)\Omega^\dagger(x)\Omega(x)\chi(x)$$
$$= \Omega(y)U(y,x)\chi(x) \qquad (15)$$

This shows $U(y,x)\chi(x)$ transforms like $\chi(y)$ and as such is also well defined. This is exactly what we need for the derivative to work. Now let us choose $y = x + \epsilon$ and define the new covariant derivative as

$$D_\mu \chi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\chi(x+\epsilon) - U(x+\epsilon, x)\chi(x)\right] (16)$$

Each term in this equations has been constructed so as to transform in the same way under the gauge transformation, so that now $D_\mu(\chi(x)) \to \Omega(x) D_\mu \chi(x)$.

Therefore the covariant derivative transforms covariantly. The problem is now to find an explicit form for D_μ for a Lagrangian which will be invariant under local symmetries. As we know, the parallel transport $U(y,x)$ is given by [6]

$$U(y,x) = \exp(ig \int_\Gamma A_\mu(x) dx^\mu)$$ where Γ is a path joining y to x. (17)

Since $U(y,x) \in SU(N)$ and therefore, $A_\mu(x)$ is an element of the Lie algebra of SU(N) with coupling g. Choosing $y = x + \varepsilon$, we get for infinitesimal path Γ from x to $y = x + \varepsilon$

$$U(x+\varepsilon, x) \approx \exp(ig\epsilon A_\mu(x))$$
$$\approx 1 + ig\epsilon A_\mu(x) + O(\epsilon^2) \qquad (18)$$

Substituting for $U(x + \in, x)$ into Eq.(16), we get

$$D_\mu \chi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\chi(x+\epsilon) - U(x+\epsilon, x)\chi(x)\right]$$
$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\chi(x+\epsilon) - \left[1 + ig\epsilon A_\mu(x) + O(\epsilon^2)\right]\chi(x)\right]$$
$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\chi(x+\epsilon) - \chi(x) + ig\epsilon A_\mu(x)\chi(x) + O(\epsilon^2)\chi(x)\right]$$
$$= \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon}\left[\chi(x+\epsilon n) - \chi(x)\right] + igA_\mu(x)n^\mu\chi(x) + O(\epsilon)\chi(x)\right)$$
$$= \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon}\left[\chi(x+\epsilon n) - \chi(x)\right] + igA_\mu(x)n^\mu\chi(x) + O(\epsilon)\chi(x)\right)$$
$$= \left[\partial_\mu - igA_\mu(x)\right]\chi(x) \qquad (19)$$

Therefore, the covariant derivative $D_\mu = \partial_\mu - igA_\mu(x)$ where $A_\mu(x)$ is an element of the Lie algebra of SU(N). The transformation properties of $A_\mu(x)$ can be found as follows

Substituting $y = x + \varepsilon$ in Eq. (14), we get

$$U(x+\varepsilon, x) \to \Omega(x+\varepsilon)U(x+\varepsilon, x)\Omega^\dagger(x)$$

But from Eq.(18), we have

$$U(x+\varepsilon, x) \approx \exp\left(ig\epsilon A_\mu(x)\right)$$
$$\approx 1 + ig\epsilon A_\mu(x)n^\mu + O(\epsilon^2)$$

Therefore

$$1 + ig\epsilon A_\mu(x)n^\mu + O(\epsilon^2)$$
$$\to \Omega(x+\varepsilon)\left[1 + ig\epsilon A_\mu(x)n^\mu + O(\epsilon^2)\right]\Omega^\dagger(x)$$

Using expansion $\Omega(x+\varepsilon) = \Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)$

$$1 + ig\epsilon A_\mu(x) + O(\epsilon^2)$$
$$\to \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right)\left[1 + ig\epsilon A_\mu(x) + O(\epsilon^2)\right]\Omega^\dagger(x)$$
$$= \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right)\Omega^\dagger(x) + \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right) ig\epsilon A_\mu(x)\Omega^\dagger(x) + \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right) O(\epsilon^2)\Omega^\dagger(x)$$
$$= \Omega(x)\Omega^\dagger(x) + \varepsilon\,\partial_\mu\Omega(x)\Omega^\dagger(x) + O(\epsilon^2)\Omega^\dagger(x) + \left(\Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + \varepsilon\,\partial_\mu\Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + O(\epsilon^2)\, ig\epsilon A_\mu(x)\Omega^\dagger(x)\right) + \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right) O(\epsilon^2)\Omega^\dagger(x)$$
$$= 1 + \varepsilon\,\partial_\mu\Omega(x)\Omega^\dagger(x) + \Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + \varepsilon\,\partial_\mu\Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + O(\epsilon^2)$$

Neglecting $O(\epsilon^2)$ on either side and cancelling 1, we get

$$ig\epsilon A_\mu(x) \to \epsilon\,\partial_\mu\Omega(x)\Omega^\dagger(x)(1 + \varepsilon\, igA_\mu(x)) + ig\epsilon\Omega(x)A_\mu(x)\Omega^\dagger(x)$$

Dividing on either side by $ig\epsilon$ and taking limit $\epsilon \to 0$, we get

$$A_\mu(x) \to -\frac{i}{g}\partial_\mu\Omega(x)\Omega^\dagger(x) + \Omega(x)A_\mu(x)\Omega^\dagger(x)$$

Rearranging we can write the required transformation equation for $A_\mu(x)$

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega^\dagger(x) - \frac{i}{g}\partial_\mu\Omega(x)\Omega^\dagger(x) (20)$$

This is precisely the transformation to make the derivative D_μ transform covariantly

$$D_\mu\chi(x) \longrightarrow \Omega(x)D_\mu\chi(x)$$

Finally we have a Lagrangian

$$\mathcal{L} = \bar{\chi}(x)\left[(i\gamma^\mu D_\mu - m)I_{N\times N}\right]\chi(x) \qquad (21)$$

which is now invariant under the local transformations of the fields

CONCLUSION

Gauge symmetries are the local symmetries which depend on space-time and therefore cause the ordinary derivative lose its meaning. This requires the quest for a covariant derivative which differs from the ordinary derivative by a term responsible for the interactions. Abelian gauge transformation belonging to a one-dimensional group explains the electromagnetic interactions via gauge fields namely photons. Quarks are the fields of Quantum Chromo dynamics (QCD) due to the non-abelian gauge transformations belonging to SU(3) group . Standard model indicates the production of the massive gauge fields $W_\mu^\pm, Z_\mu^0$ of electroweak interactions as a result of spontaneous symmetry breaking. Gauge theories thus, develop as a natural consequence of the invariance of Lagrangian under gauge transformations.

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