Functional Analytic Approach to a Classical Problem of Filtering Theory

[*1]M.Thiyagarajan, [2]K.Sivakami

[*1]Dean Research, Nehru Group of Institutions, Nehru Institute of Engineering and Technology, Coimbatore-641 105,Tamilnadu, India.
[2]Assistant Professor, Electronics and Communication Engineering, Nehru Institute of Engineering and Technology,Coimbatore-641 105,Tamilnadu, India.

[*1] m_thiyagarajan@yahoo.com, [2] ksivakaamii@gmail.com

Abstract:- Filtering theory has been developed out of systematic study of one particularly important type of analytical representation namely, the representation in terms of past innovations. We present a model for the expected value the signal given the fast of the observations upto the present time where the noise is a standard Brownian motion process. The classical result of Benes and its generalization is studied through square integrable functions of Hilbert space. The innovation equivalence theorem leads to the convergence of adaptive process of the signal. Illustration is given one dimensional random process with uncorrelated increments. The computational part employs mat lab coding and the output shows the estimation of signal from the observation.

Keywords: *Gaussian noise, Filtering theory, Innovation Equivalence theorem, Hilbert Space, Adaptive process.*

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1. Introduction

Clark has shown the measurable version of innovation process under the condition of square integrable signal. Frost and keileth give a basic result signal is a weiner process with respect to observations. This result on classical innovation problem has been settled in affirmative by clark and benes.schmidt operator was used to prove the extension of this result under fuctional analytic tools. A detailed proof is given here for these two classical results.

It is a classical problem of the filtering theory to estimate signals corrupted by noise from the past of observations. We use the notation of equivalence of two processes having the same type innovation processes, as given by rosanow (1977). We have generalized the innovations equivalence theorem of benes(1981) by using Hilbert space Theory.

2. Basic Concepts and results

Multipliants theory as we know it today is concerned with a very broad class of stochastic processes, and has developed out of a systematic study of one particularly important type of analytical representation namely, the representation in terms of past innovations. Mathematical idealization of a classical problem of filtering theory in the estimation of signals is as follows:

A. Innovation Process:

Let the signal z_t be a measurable stochastic process, with E($lz_t l$) $<\infty$ and the noise W_t a Brownian motion process. The observations consist of the process Y_t, given by

$$Y_t = \int_0^t z_s \, d_s + W_t, t > 0 \quad \dots (1)$$

Introduce

$$z_t = E(z_t \mid Y_S, \; 0 \le S \le t),$$

the expected value of z_t given the past of the observations upto it.

If

$$\int_0^t Z^2_S \, ds < \alpha \quad \text{almost surely,}$$

then there is a measurable version of

$\wedge$

Z with

$$\int_0^t Z^2_S \, ds < \alpha \quad \text{almost surely.}$$

The innovations process for this set up is defined to be.

$$\gamma_t = (\int_o^t (\mathbf{Z}_s - \hat{\mathbf{Z}}_s) \, ds) + W_{t,\ t10.}$$

B. Adopted Process

If γ_t is a weiner process with respect to observations, then

$$\mathbf{Y}_t = \int_o^t \mathbf{Z_t} \, ds + \gamma_t \quad \text{... (2)}$$

Then Z_t is adopted to Y_t

c. Innovation Problem

The innovations problem is to determine where the innovations process γ contains the same information as the observations Y_t, that is, when Y_t is adapted by γ.

3.Counter Example for innovation problem

Suppose the signal $\mathbf{Z_t}$ is a causal function a (t, y) of the observations, that

$\wedge$

is, signal is entirely determined by feedback from the observations. Then Z = Z, W= γ, the problem reduces to asking whether the observations are well defined in the strong sense of being adapted to noise; for this degenerate case the noise is the only process left. The example consists of a choice of a(.,.) for which there is a unique weak solution Y, non—anticipative in the sense that the future increments of W are independent of the past of Y, but such that Y cannot be expressed as a causal functional of W.

4. Affirmative cases of innovation problem

Affirmative cases I

Clark(1969) showed that if the noise and signal are independent and the signal is bounded (uniformly in t and w), then observations are adapted to innovations.

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Affirmative cases II

the case of Gaussian

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observations turns out affirmatively; here Z is a linear functional of Y and

the equation (2) is solvable for **Y** by a Neumann series. More generally, the

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case in which **Z** is a lip functional of **Y** also turns out affirmatively; in practice, though it is difficult to find out when this condition is met.

5. Main result

Benes-innovations Equivalence Theorem

We assume that the signal and noise are independent and the signal is square integrable almost surely.

Let the σ - algebra generated by random variables $\{ X_t \}$ tϵT where T is some indexing set, be denoted by

$$\sigma \{ X_t , t\epsilon T \}$$

Definition 1. (Kallianpur and Striebel 1968) ,

Introduce a functional q defined for pairs of suitable functions f , g by

$$q(f, g) = \exp \left\{ \int_o^t f(s)\, dg(s) — Y2 \int_o^t (f (s))^2\, ds \right\}$$

where f is a locally square integrable function and g is a wiener

function. For such pairs the stochastic integration in q is well-defined. Define

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the conditional expection Z_t as A

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$$z_t = E(Z_t \mid Y_s\, 0 \leq S \leq t)$$

$$= \frac{\int Z(d(z))\, Z_t q(z, Y)_t}{\int Z(dz) q(z,y)}$$

$$= \alpha\, (t, y)$$

Where Z is the measure for the signal process. In addition the A transformation

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$\wedge$

Z(.) is defined by

$$\hat{Z}(f)_t = \alpha (t, f) .$$

Theorem 1.

If Z and W are independent, if $E(||Z_t|) < \infty$ for each t_s, and

if p $\{ \int_0^T Z^2_S \, ds < \alpha\} = 1$ then

$\alpha(Y_s , s \leq t) = \sigma((Y_\& , s<t)$

for $O \leq t \leq T$, modulo null sets.

Proofs : We exhibit a sequence of γ adepted process converging to

$\wedge$

Z; the result follow from equation (2).

Let

$$||z|| = \int_0^T Z^2_S \, ds$$

we have the approximations

$$\frac{\int_0 Z\,(dz)\, q\,(z, y)_t\, Z_t \quad ||z||^2 \wedge |Zt| \leq m}{\int_2 Z(dz)\, q\,(z, y)_t} = \hat{Z}_m(t)$$

$$||z||^2 \wedge |Zt| \leq m \qquad = \wedge \; Z_m(Y)t$$

$\wedge$

approach Z_t as m α, ~~and~~ $\rightarrow$ also each one is adapted to Y as a

function of (t,w). Therefore it is enough to prove that each approximation above is adapted to γ as a function of (t,w).

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Set

$$M(t, m) = (\,|\,|Z|\,|^2 \wedge |Z_t| \leq m)$$

and put also

$$\hat{Z}^m{}_t, O = 0 \qquad m=1, 2, \ldots\ldots$$

$$\hat{Z}^m{}_t, n+1 = \frac{\int Z\,(dz)q\,(z, y^{mn})_t\ Z_t}{\int Z\,(d(z)q\,(z, y^{mn})_t} \qquad \max(m, n) \geq 1$$

$$\hat{Y}_n^{m,n} = \int_0^t \quad Z_s^{m,n} \ \ ds +$$

$$Y_t^m = \int_o^t \hat{z}_{m\ (S)}\ \ ds + \gamma_t$$

and note that

$$\hat{Z}\ \ m, n+1 = \hat{Z}_m\,(Y^{m,n})$$

$$d\,(Y^{m,n} - Y^m) = (\hat{Z}^{m,n} - \hat{z}^m)\ \ ds, \text{ and}$$

$$q(Z, Y^{m,n})_t = q\,(Z, Y^m)_t \exp \int_o^t Z_s\ (\hat{z}^{m,n} - \hat{Z}m)_s\ ds$$

we now set

$$\hat{Z}^{m,n} - \hat{z}^m = \gamma^{m,n}$$

and use (4) to get

$$|4_t^{m,n+1}| \leq \frac{m}{2} \int_M \int_{(t,m)} Z^2\,(dz \times d3)\ \{\int_o^t /Zs - 3s|^2\}\ ds\}^{½}$$

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$$\{ \int_o^t /\gamma_s{}^{mn} |^2 ds\}^{½}$$

$$x = \frac{q(z,Ym,n)t\, q(3,Ym)\, t+q\,(z,Ymn)t;}{\int z(dz)\, q(z,Ym)t \int z(d(3)q\,(3,Ymn)t} \quad q\,(3, Y^{mn})_t$$

M(t,m) M(t,m)

$\{\int_o^t |x^{mn}|^2$ ds $\}½$ is a factor of the right hand side, the value of the integrel over $M(t,m)^2$ with respect to Z^2 does not exceed

$2\sqrt{m}.$ This is because on $M(t,m)^2$

$$\int_o^t |z_s - 3_s|^2 \text{ ds } \leq 2 \int_o^t |x_s{}^{mn}|^2 \text{ ds}$$

Thus squaring

$$|\chi_t\, m,n+1|^2 \leq m^3 \int_o^t / x_s{}^{mn} |^2 ds$$

Jense's inequality and (4) give

$$\int_o^t |\Psi_s^{m,o}|^2 \text{ ds } = \int_o^t |\hat{-Z}_m|^2 \text{ ds} \leq m^2 t$$

Thus $\hat{z}^{m,n}$ are $\hat{\gamma}$ - adapted functions converging to z_m in probabilithy, these in turn

converge to $\hat{z}$; and hence Z is γ - adapted and the theorem is proved. We note

that the proof works for any T so long

as $\int_O^T Z_s^2$ ds $< \infty$ a.s.

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5. Innovation equivalence Theorem in Hilbert space

Definition 2. Innovation process

Let Ψ (t), $t_o < t < T$ be a one dimensional random process with uncorrelated increments. We assume that the process is left continuous

$$h^{Lt} \longrightarrow 0+ \ \Psi (t-h) \quad = \Psi (t)$$

and set

$$F (t) = E \mid \ (t) \ \mid^2$$

we shall call the monotine – non-decreasing left continuous function

$$F (t),\ t_o < t < T$$

a structural function.

The space H_t (Ψ) consists of all variables represent-able as the stochastic integral

$$\eta = \int_{to}^{t} C(S) \ \ d \ \varsigma (S) \ ,$$

Where the complex-valued function c(t), $t_o \leq t \leq T$ satisfies the condition

$$\int_{to}^{T} |C\ (t) \ |^2 \ d\,F\,(t) \ < \ \infty$$

Lect C denote the Hilbert space of all such function with the scalar product

$$\int_0^T C1\ (t) \quad \overline{C_2\ (t) \qquad\qquad} \qquad dF\ (t)$$

Let C_t be the set of all functions vanishing outside the interval (t_o, t) – we can see that the subspaces H_t ( ) are isomorphic to an appropriate subspaces C_t such that H_t ( ) consists of all variable of the form

$$\eta \ = \ \int_{to}^{t} C\ (S) \ \ d \ \ (S)$$

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where

$$\int_{to}^{t} |C(S)|^2 \ d \ F \ (S) \ < \ \infty$$

we shall call the vandom processes η (t) with uncorrelated increments satisfying

$$H_t(\eta) = H_t(\xi), \ t_o \le t \le T$$

an innovation process for the random process (t) t_o t T.

Definition 3. Equivalence

Let ξ (t) = {ς (t) , x} x ε R and

η (t) = {n (t), x} x ε R

be two random processes on a interval (t_o, T), where the parameter runs through some set R. Consider the mapping

A: {η(t), x} → {ξ (t), x} x ε R,

t belongs (t_o, T).

we shall say that the process ξ (t) is equivalent to η (t) on the interval (t_o, T) if the mapping extends to a linear bounded invetible operator A on the Hilbert space H (η) into the Hilbert space H (ξ) and in addition if the difference I-A* A is a Hilbert – schimidt operator. If the correlation operator B = A* A is invertible and the difference I-B is a Hilbert-Schimidt operator, then the families will be isometric and consequently the equivalent processes ξ (t) and η(t) $t_o < t < T$, have innovation processes of the same type. Note that for Gaussian random processes ξ(t) and η(t) $t_o < t < T$ it implies the equivalence of their probability distributions P_ξ and P_η on any function space.

Using the notion of equivalence of two processes in the Hilbert space, we prove the following innovation equivalence theorem.

Theorem 2.

For equivalence of the random processes (t) and (t) on (t_o , T) it is necessary and sufficient that the difference

$$b(s, t) = B\eta(s, t) - B\xi(s, t)$$

be absolutely continuous with respect to ds. dt. More precisely

$b(s, t) + \int_{to}^{s}\int_{to}^{t} k(s'mt')\, ds'\, dt' \quad t_o < s, <T, t_o < t < T,$

where derivative $K(s, t) = \dfrac{\partial^2 b(s, t)}{\partial s \, \partial t}$ has the square

integrable trace norm:

$$\int_{to}^{T} \int_{to}^{T} |k(s, t)|^2 \, ds \, dt < \alpha$$

thekernel K(s, t) as an operator in L^2 (R) has no eigen value equal to one.

Proof: The wiener process η (t), $t_0 < t < T$ with components

{η (t), x} xεR′ R some Hilbert space.

The natural isometry

$\{\eta(t), x\}\ x_t\ (s)\ x$

between the variables {η (t), x} H (t) and the elements γ_t (s) x belongs to L^2 (R) allows us to identify H_t (η) with subspaces L_t^2 (R), to t_o t T; here the scalar function γ_t (s) is the indicator of the interval (t_o, T), L^2 (R)is the Hilbert space of all measurable R-valued functions u(s), $t_o < s < t$, square integrable with the scalar product

(u, v) $= \int_{to}^{T} (u(s), v(s))\, ds$, u, v ε L^2 (R) is the subspace of all functions u (s) ε L^2 (R), such that u (s) = o with s < t. We assume that the standard wiener process (t), $t_o < t > T$ has the normalized correlation function

$B\eta (s, t) = \text{minimum} \{(s-s_o, t - t_o), I, \}$ … (2)

$t_o < s, <T, t_o < t < T.$

Let ξ (t) be some random process with components (ξ (t) , x), $t_o < t < T$, x ε R, and Bξ (s, t) be its correlation function in a Hilbert space R;

$E\{\xi|s), x\}\{\xi(t), Y\}$

$= \{B_\xi (s, t)\, x, Y\}$ x, Y ε R … (3)

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The family of subspaces H_t (ξ) will be isometric to the family L_t^2 (R), if, the process ξ (t) is equivalent to the wiener process η(t). In this case the equivalence implies that the operator A

$$A : \ \chi(s)\ x \longrightarrow \{\xi(t), x\}$$

defined on the whole (L^2(R)) system of elements

$$u(s) = \chi(s)\ x$$

can be continued upto a linear bounded invertible operator A, such that the difference I-A* A is a Hilbert – Schmidt operator on the function space L^2 (R)

But any Hilber-schmidt operator in the function space L^2 (RxR) by the operator function in the Hilbert space R with values K(s, t) and with square-integrable trace norm

$$\int_{to}^{T} \int_{to}^{T} |k(s, t)|^2 \ ds\ dt < \infty$$

where

$$|k(s, t)|^2 = sp\ (k(s, t)^* \ .\ K(s, t))$$

consequently, under the equivalence condition (u, v) – (Au, AV)

$$= \int_{to}^{T} \int_{to}^{T} k(s, t)\ u(s),\ v(t) \ \ ds\ dt$$

where k (s, t) is some kernel in L^2 (R x R).

The relation (7) wil be satisfied if it is satisfied for some complete system of elements with u (s) = γ_{t1} (s) x, V(t) = γ_{t2} (t) y the relation can be represented in an operator form:

$$B\eta\,(t_1 , t_2) – B\,\xi(t_1 , t_2)$$

$$= \int_{to}^{T} \int_{to}^{T} k(s, t)\ ds\ dt$$

If the representation holds, then the corresponding operator I-A* A will be a Hilbert-Schmidt operator.

Assume B = A* and F = I-B. The condition of boundedness and invertibility of operator H is equivalent to the fact that the operator B possesses the same properties. If F is a Hilbert-schmidt operator the B will be a bounded operator, and the condition of invertibility of the positive operator B:

$$\inf_{||u||=1} (Bu, u) = 1 - \sup_{||u||=1} (Fu, u) \; 0$$

is equivalent to the fact that the self-adjoint operator F has no eigen values equal to 1- its maximal eigen value is less than1.

This completes the proof the theorem.

Illustration.

Let η (t), t ε (t_o , T) be a one dimensional random process with uncorrelated increments and

$F(t) = E|\eta(t)|^2$, t ε (to , T) be its structural function.

Let a (t), t ε (t_o , T) be a random process such that

$$\int_{to}^{t} E|a(t)|^2 \, dt < \alpha$$

let $\xi(t) = \int_{to}^{t} a(s) \, d\, F(s) + \eta(t).$

Remembering that the projection of the variable a (t) onto the subspace $H_t(\eta)$ is representable as the stochastic integral

$$\int_{o}^{t} c(u, v) \, d\eta(v)$$

where the integrand satisfies the condition

$$\int_{to}^{t} |C(u, v)|^2 \, d\, F(v) \le E\,|a(u)|^2$$

Now we obtain

$$E\,(a(t)\;\eta(t)) = E\,(\int_{to}^{t} C\overline{(u, v)\;\; d\eta}(v) \int_{to}^{t} d\eta(u))$$

$$= E\,(\int_{to}^{t} C(u, v)\;\; d\;F(V))$$

Therefore We have

$$B_{\dot\eta}(s,t) - B_{\xi}(s,t) = \int_{to}^{t} \int_{t0}^{s} k(u,v)\, dF(u)\, dF(v)$$

Then

$$K(u,v) = - C(u,v) - \overline{C(u,v)}$$

$$= - E(a(u)\,\overline{a(u)})$$

Thus the condition of Theoram 2() is true, and the random process ξ (t) is equivalent to $\dot{\eta}$ (t) if the operator

A: $\sum C_k \dot{\eta} (t_k) \to \sum C_k \xi (t_k)$ is bounded and invertible.

Under the condition the process $\dot{\eta}$ (t) is the innovation process for ξ (t). The condition holds good, for instant, if the processes a(t) and $\dot{\eta}$ (t) are orthogonal to each other. Indeed, in this case

$K(s,t) = - E(a(s)\overline{a(t)})$ is negative definite.

$$\int_{to}^{t}\int_{to}^{t} K(s,t)\, u(s)\, \overline{u(t)\, dF}(s)\, dF(t)$$

$$= -E \left|\int_{to}^{t} a(t)\, U(t) dF(t)\right|^2 < \infty$$

For any non zero function u(t) such that $\int_0^t |u(t)|^2\, dF(t) < \infty$

Using the above illustration we can conclude that, if the operator

A: $\sum C_k\ \dot{\eta} (t_k) \to \sum C_k\, \xi (t_k)$

Is bounded and invertible;

ξ and $\dot{\eta}$ are equivalent. One such case is the instance in which a(t) and $\dot{\eta}$ (t) are orthogonal and in this case we get Benes condition namely E(a(t) a(t)) is positive definite.

6. Conclusion

Innovation Process plays an important role in the information theory of filtering and communication. The major results of benes and clark have been presented in detail using functional analytic approach. Appropriate illustrations are given to understand clearly with major contributions in this area. Real time problems are being attempted in the area of signals and systems in identifying true signal from finite observations obeying innovations equivalence.

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