Some Properties on Strong Roman Domination in Graphs

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***Abstract*—**A *Strong Roman dominating function* (SRDF) is a function $f:V\to\{0,1,2,3\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=3$ and every vertex u for which $f(u)=1$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of an SRDF is the value $f(V)=\sum_{u\in V} f(u)$. The minimum weight of an SRDF on a graph G is called the *Strong Roman domination number* of G. In this paper, we attempt to verify some properties on SRDF and moreover we present Strong Roman domination number for some special classes of graphs. Also we show that for a tree T with $n\geq 3$ vertices, l leaves and s support vertices, we have $\gamma_{SR}(T)\leq \frac{6n-l-s}{4}$ and we characterize all trees achieving this bound.

Keywords*-Roman domination number, Strong Roman domination number, Graph, Tree, Star, Double star, Connected graph.*

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I. Introduction

Mathematical study of domination in graphs began around 1960, there are some references to domination related problems about 100 years prior. In 1862, De Jaenisch [2] attempted to determine the minimum number of queens required to cover a $n\times n$ chess board. Except as indicated otherwise, all terminology and notation follows [5, 4, 9]. Let $G=(V,E)$ be a graph of order $|V|=n$. For any vertex $v\in V$ the *open neighborhood* of v is the set $N(v)=\{u\in V \mid uv\in E\}$ and the *closed neighborhood* is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$ the open neighborhood of S is $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S]=N(S)\cup S$. A set S of vertices is called a *vertex cover* if for every edge $uv\in E$ either $u\in S$ or $v\in S$. A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected.* A graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. A *rooted tree* T distinguishes a vertex r called the *root*. A vertex of degree 1 is called a *leaf* which denoted by l. A *adjacent leaf* of vertex u in a tree T is a neighborhood of u that is a leaf in T. A *support vertex* (also called a *stem* in the literature) is a vertex of degree at least 2 that is adjacent to at least one leaf. A support vertex adjacent to two or more leaves is a *Strong support vertex*. A *Weak support vertex* is a support vertex that is adjacent to exactly one leaf. Also we denote the set of leaves in G by $L(G)$ and the setof support vertices by S(G). A *Star* is the graph $K_{1,k}$ where $k\geq 1$. If $k>1$, the vertex of degree k is called the C*enter vertex* of the star. A *Double star* is formed from two disjoint stars by joining the center vertices of each by an edge. Thus a Double star is a tree with exactly two vertices that are not leaves.

We now introduce the concept of dominating sets in graphs. A set $S\subseteq V$ is a *dominating set* if $N[S]=V$ or equivalently, every vertex in $V-S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G and a dominating set S of minimum cardinality is called a $\gamma(G)-$*set* of G, see [10]. Let $f:V\to\{0,1,2\}$ be a function having the property that for every vertex $v\in V$ with $f(v)=0$, there exists a neighborhood $u\in N(v)$ with $f(u)=2$. Such a function is called a *Roman dominating function* or just an RDF. The weight of an RDF is the value $f(V)=\sum_{u\in V} f(u)$. The minimum weight of an RDF on

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G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$, see [1, 10].

K. Selvakumar et al. [8] introduced Strong Roman domination in 2016. A *Strong Roman dominating function* (SRDF) is a function $f: V \to \{0,1,2,3\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=3$ and every vertex u for which $f(u)=1$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of an SRDF is the value $f(V)=\sum_{u\in V} f(u)$. The minimum weight of an SRDF on a graph G is called the *Strong Roman domination number*of G.

In 2004, Cockayne et al. [1] studied the graph theoretic properties of Roman dominating sets. In recent years several authors studied the concept of Roman dominating functions and Roman domination numbers [12, 6, 7, 11]. In this paper, we present some results on SRDF and Strong Roman domination number for some special classes of graphs. Also we show that for a tree T with $n \ge 3$ vertices, l leaves and s support vertices, $\gamma_{SR}(T) \le \frac{6n-l-s}{4}$ and we characterize all trees achieving this bound.

Proposition: For any graph G, there exists an SRDF, $f=(V_0,V_1,V_2,V_3)$ of G, such that $V_1=\phi$.

Proof:

Let $V_1=\phi$ and $u\in V$. By the definition of SRDF, there exists a vertex $v\in V_2$ such that $v\in N(u)$. Hence a function $g=(V_0\cup\{u\}, V_1-\{u\}, V_2-\{v\}, V_3\cup\{v\})$ is an SRDF. Continuing with the same argument we find an SRDF with $V_1=\phi$. Therefore, the proposition follows. □

Theorem 1: For any graph G,

$$2\gamma(G)\le\gamma_{SR}(G)\le 3\gamma(G).$$

Proof:

Suppose that $f=(V_0,V_1,V_2,V_3)$ is a $\gamma_{SR}(G)-function$ and $|V_0|=n_0$, $|V_1|=n_1$, $|V_2|=n_2$ and $|V_3|=n_3$.

$$\begin{aligned}\gamma_{SR}(G) &= f(V)\\ &= \sum_{u\in V} f(u)\\ &= 3n_3+2n_2+n_1.\end{aligned}$$

$V_3 > V_0 \to$ The set V_3 dominates the set V_0.

$V_2 > V_1 \to$ The set V_2 dominates the set V_1.

It is implied that $V_2\cup V_3$ is a dominating set of G. So $\gamma(G)\le|V_2|+|V_3|$, thus

$$\begin{aligned}2\gamma(G) &\le 2|V_2|+2|V_3|\\ &\le |V_1|+2|V_2|+3|V_3|\\ &= \gamma_{SR}(G).\end{aligned}$$

Hence

$$2\gamma(G)\le\gamma_{SR}(G). \qquad (1)$$

Now, let S be a $\gamma-set$ of G. Then $\gamma(G)=|S|$. We can define an SRDFon G, for all $v\in S$ we have $f(v)=3$ and also for all $u\notin S$ we have $f(u)=0$. Therefore $(V_0,V_1,V_2,V_3)=(\phi,\phi,\phi,S)$ is an SRDF. It is impliedthat $|V_0|=0$, $|V_1|=0$, $|V_2|=0$ and $|V_3|=|S|$. Therefore

$$\begin{aligned}\gamma_{SR}(G) &\le 3|V_3|\\ &= 3|S|\\ &= 3\gamma(G).\end{aligned}$$

Hence

$$\gamma_{SR}(G)\le 3\gamma(G). \qquad (2)$$

From (1) and (2) we get $2\gamma(G)\le\gamma_{SR}(G)\le 3\gamma(G)$. □

Theorem 2: For any graph G of order n,

$$\gamma_{SR}(G)=2\gamma(G)$$

if and only if $G=\overline{K}_n$.

Proof:

Suppose that $f=(V_0,V_1,V_2,V_3)$ isa $\gamma_{SR}(G)-function$. Thus $V_2\cup V_3$ is a dominating set of the graph G. Therefore $\gamma(G)\le|V_2|+|V_3|$. The equality $\gamma_{SR}(G)=2\gamma(G)$ implies that we have equality in

$$\begin{aligned}2\gamma(G) &\le 2|V_2|+2|V_3|\\ &= 2|V_2|+3|V_3|\\ &= \gamma_{SR}(G).\end{aligned}$$

So $|V_3|=0$, which implies that $V_0=\phi$. Hence all vertices are assigned with 2 and therefore

$$\begin{aligned}\gamma_{SR}(G) &= 2|V_2|\\ &= 2n.\end{aligned}$$

This implies that $\gamma(G)=n$ which shows that $G=\overline{K}_n$.

Conversely, It is obvious that if $G=\overline{K}_n$, then $\gamma_{SR}(G)=2\gamma(G)$. □

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Theorem 3: For any graph G,

$$\gamma_R(G) \neq \gamma_{SR}(G).$$

Proof:

Suppose that $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{SR}(G)-$ *function*. Let $g = (Y_0, Y_1, Y_2)$ be an RDF on G where $Y_0 = V_0$, $Y_1 = V_2$ and $Y_2 = V_3$. Therefore

$$\begin{aligned}\gamma_R(G) &\leq |Y_1| + 2|Y_2| \\ &= |V_2| + 2|V_3| \\ &< 2|V_2| + 3|V_3| \\ &= \gamma_{SR}(G).\end{aligned}$$

Thus $\gamma_R(G) \neq \gamma_{SR}(G)$. □

Based on above theorem, we know that $\gamma_{SR}(G) \geq \gamma_R(G)+1$. In the next theorem, we will discuss the equation of this inequality.

Theorem 4: $\gamma_{SR}(G) = \gamma_R(G)+1$ if and only if $\Delta(G) = n-1$.

Proof:

Suppose that $\gamma_{SR}(G) = \gamma_R(G)+1$ and $f = (V_0, V_1, V_2, V_3)$ isa $\gamma_{SR}(G)-$ *function*. Define $g = (Y_0, Y_1, Y_2)$ is an RDF on G where $Y_0 = V_0$, $Y_1 = V_2$ and $Y_2 = V_3$. Therefore

$$\begin{aligned}\gamma_R(G) &\leq |Y_1| + 2|Y_2| \\ &= |V_2| + 2|V_3|.\end{aligned}$$

On the other hand, if $|V_2| \neq 0$ and also $|V_3| \neq 0$, then

$$\begin{aligned}|V_2| + 2|V_3| &\leq |V_2| + 2|V_3| + (|V_2|-1) + (|V_3|-1) \\ &= 2|V_2| + 3|V_3| - 2.\end{aligned}$$

Hence

$$\begin{aligned}\gamma_R(G)+1 &\leq |V_2| + 2|V_3| + 1 \\ &\leq 2|V_2| + 3|V_3| - 1 \\ &= \gamma_{SR}(G) - 1.\end{aligned}$$

Thus $\gamma_R(G)+1 \neq \gamma_{SR}(G)$, which is a contradiction. Therefore $|V_3| = 0$ or $|V_2| = 0$.

Let $|V_3| = 0$ and if $|V_2| > 1$, then

$$\begin{aligned}\gamma_R(G) &\leq |V_2| \\ &\leq 2|V_2| - 2.\end{aligned}$$

Hence

$$\begin{aligned}\gamma_R(G)+1 &\leq |V_2| + 1 \\ &\leq 2|V_2| - 1 \\ &= \gamma_{SR}(G) - 1.\end{aligned}$$

Thus $\gamma_R(G)+1 \neq \gamma_{SR}(G)$ which is a contradiction. Therefore in this case, $|V_2| = 1$ and thus $G = K_1$.

Now, assume that $|V_2| = 0$ and if $|V_3| > 1$, then

$$\begin{aligned}\gamma_R(G) &\leq 2|V_3| \\ &\leq 3|V_3| - 2.\end{aligned}$$

Similarly, we get $\gamma_R(G)+1 \neq \gamma_{SR}(G)$, which is a contradiction. Therefore in this case, $|V_3| = 1$ and if $V_3 = \{v\}$, then $\deg(v) = n-1$. Thus G has a vertex of degree $n-1$.

Therefore in each case $\Delta(G) = n-1$.

Conversely, If $\Delta(G) = n-1$, then $\gamma_{SR}(G) = 3$ and $\gamma_R(G) = 2$. So $\gamma_{SR}(G) = \gamma_R(G)+1$. □

Theorem 5: For any path P_n,

$$\gamma_{SR}(P_n) = \begin{cases} n & , \quad n \equiv 0 \pmod 3 \\ n+1 & , \quad n \not\equiv 0 \pmod 3 \end{cases}.$$

Proof:

Suppose that a, b and c are consecutive vertices and $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{SR}(G)-$ *function* of P_n, respectively. If two vertices of $\{a, b, c\}$ belongingto V_0, then either one of those vertices belongs to V_3, which in this case we have

$$f(a) + f(b) + f(c) \geq 3,$$

or $a, c \in V_0$ and $b \in V_2$. In this case, all vertices which are adjacent to a and c are named x and y should belongto V_3. Therefore

$$f(x) + f(a) + f(b) + f(c) + f(y) \geq 8.$$

So, always

$$\begin{aligned}\gamma_{SR}(P_n) &= f(V) \\ &\geq n.\end{aligned}$$

Now, we use an induction on the order n. Assume the result is true for $n \leq 6$. Suppose that $n \geq 7$ and it is true for $m < n$. If $n \equiv 0 \pmod 3$ and $P_n = v_1 v_2 \cdots v_n$, we put

$$f(v_i) = \begin{cases} 3 & , \quad i \equiv 2 \pmod 3 \\ 0 & , \quad i \not\equiv 2 \pmod 3 \end{cases}.$$

Hence, f is an SRDF and also

$$f(V)=\sum_{i=1}^{n} f(v_i)$$
$$=n.$$

Therefore $\gamma_{SR}(P_n)\le n$.

On the other hand, we have shown for any n, $\gamma_{SR}(P_n)\ge n$.

Hence $\gamma_{SR}(P_n)=n$.

Now, let $n\not\equiv 0(\mathrm{mod}\,3)$ and $f=(V_0,V_1,V_2,V_3)$ be a $\gamma_{SR}(G)-$ *function*.

If $V_2=\phi$, then it is easy to show that $\gamma_{SR}(P_n)=f(V)\ge n+1$.

If $V_2\ne\phi$, then assume that for $1\le i\le n$, $f(v_i)=2$. We consider the following cases:

Case 1: $i=1$ or $i=n$.

Without loss of generality, suppose that $i=1$, $f(v_1)=2$. Hence

$$P_n-v_1=P_{n-1}.$$

By the induction, we know that $\gamma_{SR}(P_{n-1})\ge n-1$. On the other hand, it is clearly, $g=f|_{P_{n-1}}$ be an SRDFon P_{n-1}. Thus

$$g(V)\ge\gamma_{SR}(P_{n-1})$$
$$\ge n-1.$$

Therefore

$$\gamma_{SR}(P_n)=f(V)$$
$$=g(V)+2$$
$$\ge(n-1)+2$$
$$=n+1.$$

Hence $\gamma_{SR}(P_n)\ge n+1$.

Similarly, we can prove that the result is true for $i=n$.

Case 2: $i\ne 1,n$.

Hence we put $P_{i-1}=v_1\,v_2\ldots v_{i-1}$ and $P_{n-i}=v_{i+1}\,v_{i+2}\ldots v_n$. We know that

$$\gamma_{SR}(P_{i-1})\ge i-1,$$
$$\gamma_{SR}(P_{n-i})\ge n-i.$$

On the other hand, it is clearly, the functions $g_1=f|_{P_{i-1}}$ and $g_2=f|_{P_{n-i}}$ are two SRDFs on P_{i-1} and P_{n-i}, respectively. Thus

$$g_1(V)\ge\gamma_{SR}(P_{i-1})$$
$$\ge i-1,$$
$$g_2(V)\ge\gamma_{SR}(P_{n-i})$$
$$\ge n-i.$$

Therefore

$$\gamma_{SR}(P_n)=f(V)$$
$$=g_1(V)+g_2(V)+2$$
$$\ge(i-1)+(n-i)+2$$
$$=n+1.$$

Hence $\gamma_{SR}(P_n)\ge n+1$.

Therefore in both cases $\gamma_{SR}(P_n)\ge n+1$.

Now we define the function f on path P_n if $n\not\equiv 0\,(\mathrm{mod}\,3)$ as follows:

1) If $n\equiv 1(\mathrm{mod}\,3)$, then

$$f(v_i)=\begin{cases}3 & , \quad i\equiv 2\,(\mathrm{mod}\,3)\\ 0 & , \quad i\not\equiv 2\,(\mathrm{mod}\,3), i<n.\\ 2 & , \quad i=n\end{cases}$$

2) If $n\equiv 2\,(\mathrm{mod}\,3)$, then

$$f(v_i)=\begin{cases}3 & , \quad i\equiv 2\,(\mathrm{mod}\,3)\\ 0 & , \quad i\not\equiv 2\,(\mathrm{mod}\,3)\end{cases}.$$

In both cases f is an SRDF on path P_n of weight $n+1$. Hence $\gamma_{SR}(P_n)\le n+1$.

Since we have shown that before $\gamma_{SR}(P_n)\ge n+1$, therefore if $n\not\equiv 0(\mathrm{mod}\,3)$, then we have $\gamma_{SR}(P_n)=n+1$. □

II. A NEW UPPER BOUND IN TREES

It has been shown that the domination number of a connected graph G of order n is at most $\frac{n}{2}$ [4]. Regarding the fact that $\gamma_{SR}(G)\le 3\gamma(G)$, we get

$$\gamma_{SR}(G)\le 3\gamma(G)\le\frac{3n}{2}.$$

Our aim in this section is improve this bound on trees. We show that for any tree T of order n with l leaves and s support vertices, we have $\gamma_{SR}(T)\le\frac{6n-l-s}{4}$. Moreover, we characterize all trees achieving this bound.

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Theorem 6: If T is a tree of order $n \geq 3$ with l leaves and s support vertices, then

$$\gamma_{SR}(T) \leq \frac{6n-l-s}{4}.$$

Proof:

We prove this by induction on order n of tree T.

If $diam(T)=2$, then T is a Star. Therefore $l=n-1$, $s=1$ and $\gamma_{SR}(T)=3$. Hence

$$\begin{aligned}\gamma_{SR}(T) &= 3\\ &< \frac{6n-l-s}{4}\\ &= \frac{6n-(n-1)-1}{4}\\ &= \frac{5n}{4}.\end{aligned}$$

Now, assume that $diam(T)=3$. In this case, T is a Double Star $S_{a,b}$ with central vertices u and v with degrees of a and b, respectively. Without loss of generality, suppose that $a \geq b$.

If $a=2$, then $b=2$ and thus $T=P_4$. Therefore

$$\begin{aligned}\gamma_{SR}(T) &= 5\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

Now, let $a \geq 3$. If $b=2$, then the function $f=(N(u),\phi,N(v),\{u\})$ is a $\gamma_{SR}(T)-function$. Since $n \geq 5$, $l=n-2$ and $s=2$, we have

$$\begin{aligned}\gamma_{SR}(T) &= f(V)\\ &= 2|V_2|+3|V_3|\\ &= 2|N(v)|+3\\ &= 5\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Now, let $b \geq 3$. In this case, $n \geq 6$, $l=n-2$, $s=2$ and the function $f=(V(T)-\{u,v\},\phi,\phi,\{u,v\})$ is an SRDF on tree T. Hence

$$\begin{aligned}\gamma_{SR}(T) &\leq f(V)\\ &= 2|V_2|+3|V_3|\\ &= 6\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

So, we can assume that $diam(T) \geq 4$. If T has a Strong support vertex u and also v and w are adjacent leaves to u, then we consider $T'=T-w$. Let n', l' and s' be order, number of leaves and number of vertices of tree T', respectively. Since $diam(T) \geq 4$, we get $n' \geq 3$. Therefore by induction $\gamma_{SR}(T') \leq \frac{6n'-l'-s'}{4}$. Suppose that $g=(V_0,\phi,V_2,V_3)$ is a $\gamma_{SR}(T')-function$.

If $g(u)=3$, then extension of g by assigning the weight 0 to w is an SRDF on tree T. Thus, since $l'=l-1$ and $s'=s$, we have

$$\begin{aligned}\gamma_{SR}(T) &\leq g(V)\\ &= \gamma_{SR}(T')\\ &\leq \frac{6n'-l'-s'}{4}\\ &= \frac{6(n-1)-(l-1)-s}{4}\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

Now, let $g(u) \neq 3$. Then $g(v)=2$. Therefore the function f with $f(w)=f(v)=0$, $f(u)=3$ and for any other vertex x, we have $f(x)=g(x)$ is an SRDF on tree T. Hence

$$\begin{aligned}\gamma_{SR}(T) &\leq f(V)\\ &= g(V)+1\\ &= \gamma_{SR}(T')+1\\ &\leq \frac{6n'-l'-s'}{4}+1\\ &= \frac{6(n-1)-(l-1)-s}{4}+1\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Therefore, we can consider the following Fact.

Fact: T has no Strong support vertex.

We root the tree T at vertex x_0. Support that $P=x_0\, x_1 \cdots x_d$ is a diagonal path. Based on Fact, $\deg(x_{d-1})=2$. We consider the following cases:

Case 1: $\deg(x_{d-2}) \geq 3$.

In this case, every child of x_{d-2} is either a leaf or a support vertex of degree 2. Since based on Fact, T does not has a Strong support vertex, we consider $T' = T - \{x_d, x_{d-1}\}$. So, $n' = n-2$, $l' = l-1$ and $s' = s-1$. Suppose that $f' = (V'_0, V'_1, V'_2, V'_3)$ isa $\gamma_{SR}(T')-$ *function*.

If $f'(x_{d-2}) = 2$, then the function $f = (V_0, V_1, V_2, V_3)$ where $V_0 = V'_0 \cup \{x_d, x_{d-2}\}$, $V_1 = V'_1 = \phi$, $V_2 = V'_2 - \{x_{d-2}\}$ and $V_3 = V'_3 \cup \{x_d\}$ is an SRDF on tree T. Thus

$$\begin{aligned}\gamma_{SR}(T) &\le f(V)\\ &= f'(V)+1\\ &= \gamma_{SR}(T')+1.\end{aligned}$$

Since $diam(T) \ge 4$, we get $n' \ge 3$. Then under the hypothesis

$$\begin{aligned}\gamma_{SR}(T) &\le \gamma_{SR}(T')+1\\ &\le \frac{6n'-l'-s'}{4}+1\\ &= \frac{6(n-2)-(l-1)-(s-1)}{4}+1\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Now, let $f'(x_{d-2}) = 3$. So, the function $f = (V'_0 \cup \{x_{d-1}\}, \phi, V'_2 \cup \{x_d\}, V'_3)$ is an SRDF on tree T. Thus by hypothesis we have

$$\begin{aligned}\gamma_{SR}(T) &\le f(V)\\ &= f'(V)+2\\ &= \gamma_{SR}(T')+2\\ &\le \frac{6n'-l'-s'}{4}+2\\ &= \frac{6(n-2)-(l-1)-(s-1)}{4}+2\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Therefore, we can assume that for each $\gamma_{SR}(T')-$ *function*, the weight of the vertex x_{d-2} is equals to 0.

Assume that x_{d-2} is a support vertex. Based on Fact, T has only an adjacentleaf .we consider u as an adjacent leaf to x_{d-2}. If x_{d-2} has a support child v other than x_{d-1}, then since $f'(x_{d-2}) = 0$, we can assume $f'(v) = 3$, $f'(u) = 2$ and the child weight of v is equals to 0. By changing the weight of the vertices u and v to 0, x_{d-2} to 3 and child of v to 2 a new $\gamma_{SR}(T')-$ *function* is obtained where weight of x_{d-2} is not equals to 0, which is a contradiction. Since we assume that for each $\gamma_{SR}(T')-$ *function*, we have the weight of x_{d-2} is 0. So, we can assume that the only support child x_{d-2} is the vertex x_{d-1}. We put $T' = T - T_{x_{d-2}}$. In this case, $n' = n-4$. Since $diam(T) \ge 4$, we get $n' \ge 2$.

If $n' = 2$, then $T = F_1$ shown in the figure (1). In this case, $n = 6$, $l = s = 3$ and $\gamma_{SR}(T) = 7$. Therefore

$$\begin{aligned}\gamma_{SR}(T) &= 7\\ &< \frac{30}{4}\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

Now, assume that $n' \ge 3$. Therefore based on inductive hypothesis $\gamma_{SR}(T') \le \frac{6n'-l'-s'}{4}$.

Any $\gamma_{SR}(T')-$ *function* can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-2}, 2 to x_d and 0 to x_{d-1} and u. Thus $\gamma_{SR}(T) \le \gamma_{SR}(T')+5$.

If $\deg(x_{d-3}) = 2$, then $l' = l-1$ and $s' \ge s-2$. Therefore

$$\begin{aligned}\gamma_{SR}(T) &\le \gamma_{SR}(T')+5\\ &\le \frac{6n'-l'-s'}{4}+5\\ &= \frac{6(n-4)-(l-1)-(s-2)}{4}+5\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Now, let $\deg(x_{d-3}) \ge 3$. In this case, $l' = l-2$ and $s' = s-2$. Hence

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$$\gamma_{SR}(T) \le \gamma_{SR}(T') + 5$$
$$\le \frac{6n' - l' - s'}{4} + 5$$
$$= \frac{6(n-4) - (l-2) - (s-2)}{4} + 5$$
$$= \frac{6n - l - s}{4}.$$

Therefore in this case, if x_{d-2} is a support vertex, then $\gamma_{SR}(T) \le \dfrac{6n - l - s}{4}$.

Now, assume that x_{d-2} is not a support vertex.

If x_{d-2} has three children u, v and w other than x_{d-1}, then we put $T' = T - \{x_d, x_{d-1}\}$. We already assumed that for each $\gamma_{SR}(T') - function$ the weight of x_{d-2} is equalsto 0. We consider that the function f' is a $\gamma_{SR}(T') - function$, therefore $f'(x_{d-2}) = 0$. Hence we can assume $f'(u) = f'(v) = f'(w) = 3$ and the child weight of each of the vertices u, v and w are 0. In this case, by changing the weight of the vertices u, v and w to 0, childof each of the vertices u, v and w to 3 and x_{d-2} to 3, we obtain a $\gamma_{SR}(T') - function$ where the weight of the vertex x_{d-2} is not equals to 0 which is a contradiction, since we previously assumed that for each $\gamma_{SR}(T') - function$, we have the weight of the vertex x_{d-2} is equals to 0.

So, we can assume that x_{d-2} has at most two support children other than x_{d-1}.

First, assume x_{d-2} has two support children u and v other than x_{d-1}. We put $T' = T - T_{x_{d-2}}$. Since $diam(T) \ge 4$, we get $n' \ge 2$.

If $n' = 2$, then $T = F_2$ shown in the figure (1). In this case, $n = 9$, $l = s = 4$ and $\gamma_{SR}(T) = 11$. Thus $\gamma_{SR}(T) < \dfrac{6n - l - s}{4}$.

Now, let $n' \ge 3$. In this case, $n' = n - 7$ and under the hypothesis $\gamma_{SR}(T') \le \dfrac{6n' - l' - s'}{4}$. Any $\gamma_{SR}(T') - function$ can be extended to an SRDF on tree T by assigning the weight 3 to the vertices u, v and x_{d-1} and 0 to all their neighboring vertices. Therefore $\gamma_{SR}(T) \le \gamma_{SR}(T') + 9$.

If $\deg(x_{d-3}) = 2$, then $l' = l - 2$ and $s' \ge s - 3$. Hence

$$\gamma_{SR}(T) \le \gamma_{SR}(T') + 9$$
$$\le \frac{6n' - l' - s'}{4} + 9$$
$$\le \frac{6(n-7) - (l-2) - (s-3)}{4} + 9$$
$$< \frac{6n - l - s}{4}.$$

Now, let $\deg(x_{d-3}) \ge 3$. In this case, $l' = l - 3$ and $s' = s - 3$. Thus

$$\gamma_{SR}(T) \le \gamma_{SR}(T') + 9$$
$$\le \frac{6n' - l' - s'}{4} + 9$$
$$= \frac{6(n-7) - (l-3) - (s-3)}{4} + 9$$
$$= \frac{6n - l - s}{4}.$$

Now, we support that x_{d-2} has only one support child other than x_{d-1}. Let u be a support child x_{d-2} other than x_{d-1}. We put $T' = T - T_{x_{d-2}}$. So $n' = n - 5$, $l' \ge l - 2$ and $s' \ge s - 2$. Since $diam(T) \ge 4$, we get $n' \ge 2$.

If $n' = 2$, then $T = F_3$ shown in figure (1). So, $n = 7$, $l = s = 3$ and $\gamma_{SR}(T) = 9$. Hence $\gamma_{SR}(T) = \dfrac{6n - l - s}{4}$.

Now, let $n' \ge 3$. So based on inductive hypothesis $\gamma_{SR}(T') \le \dfrac{6n' - l' - s'}{4}$. Any $\gamma_{SR}(T') - function$ can be extended to an SRDF on tree T by assigning the weight 3 to the vertices u and x_{d-1} and 0 to all their neighboring vertices. So, $\gamma_{SR}(T) \le \gamma_{SR}(T') + 6$. And therefore

$$\begin{aligned}\gamma_{SR}(T) &\le \gamma_{SR}(T')+6\\ &\le \frac{6n'-l'-s'}{4}+6\\ &\le \frac{6(n-5)-(l-2)-(s-2)}{4}+6\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

Case 2: $\deg(x_{d-2})=2$.

We put $T'=T-T_{x_{d-2}}$. Since $diam(T)\ge 4$, we get $n'\ge 2$.

If $n'=2$, then $T=P_5$. Thus

$$\gamma_{SR}(T)=5 = \frac{6n-l-s}{4}.$$

Now, let $n'\ge 3$. In this case, $n'=n-3$, $l'\ge l-1$ and $s'\ge s-1$. Any $\gamma_{SR}(T')-function$ can be extended to an SRDF on tree T by assigning the weight 3 to the vertices u and x_{d-1} and 0 to all their neighboring vertices. Therefore $\gamma_{SR}(T)\le\gamma_{SR}(T')+3$. Hence with the hypothesis we have

$$\begin{aligned}\gamma_{SR}(T) &\le \gamma_{SR}(T')+3\\ &\le \frac{6n'-l'-s'}{4}+3\\ &\le \frac{6(n-3)-(l-1)-(s-1)}{4}+3\\ &< \frac{6n-l-s}{4}.\end{aligned}$$

So the problem is solved. □

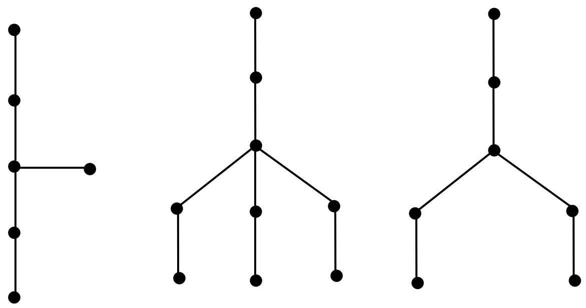

Figure 1.Trees F_1, F_2 and F_3.

In the following, we characterize all the trees subjected to the condition $\gamma_{SR}(T)=\dfrac{6n-l-s}{4}$. Let F be a family of trees T where it comes from a sequence of trees $T_1, T_2, \dots, T_j$, $(j\ge 1)$ such that $T_1=P_4$ or $T_1=F_3$ (Shown in Fig 1) and if $j\ge 2$, then T_{j+1} can be obtained from T_j with one of two operations O_1 or O_2.

A. *Operation* O_1

Let $u\in V(T_j)$, $\gamma_{SR}(T_j-u)\ge\gamma_{SR}(T_j)$ and $\deg(u)\ge$ ... this case, T_{j+1} is obtained from T_j by adding a ... with the support vertex v and adding the edge ...

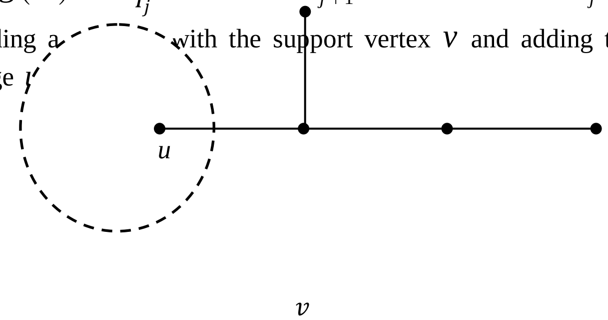

Figure 2.Operation O_1.`

B. *Operation* O_2

Let $u\in V(T_j)$, $\gamma_{SR}(T_j-u)\ge\gamma_{SR}(T_j)$ and $\deg(u)\ge 2$. In this case, T_{j+1} is obtained from T_j by adding a tree F_3 with adding the edge uv where v is a central vertex of tree F_3. (See Fig 3)

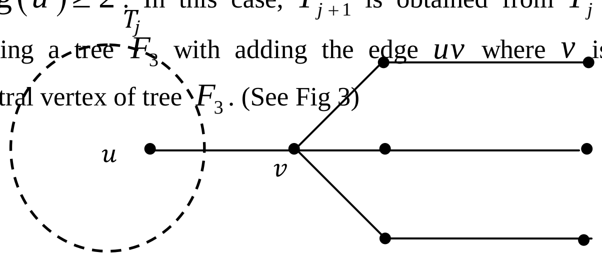

Figure 3.Operation O_2.

To prove that each tree $T\in F$ satisfy the condition $\gamma_{SR}(T)=\dfrac{6n-l-s}{4}$, the following two Lemmas will be useful. For each $k\ge 1$, let n_k, l_k and s_k denote order, number of leaves and number of support vertices of tree T_k, respectively.

Lemma 1: Let $\gamma_{SR}(T_j)=\frac{6n_j-l_j-s_j}{4}$ and T_{j+1} is obtained by T_j with operation O_1, then

$$\gamma_{SR}(T_{j+1})=\frac{6n_{j+1}-l_{j+1}-s_{j+1}}{4}.$$

Proof:

Suppose that the path $P_4=xvyz$ and the vertex u is operation dependent. Any $\gamma_{SR}(T_j)-function$ can be extended to an SRDF on tree T by assigning the weight 3 to v, 2 to z and 0 to x and y. Therefore $\gamma_{SR}(T_{j+1})\le\gamma_{SR}(T_j)+5$. Now, suppose that f is a $\gamma_{SR}(T_{j+1})-function$.

If $f(u)\neq 0$, then $f|_{T_j}$ is an SRDF on tree T_j. So, in this case, we have

$$\begin{aligned}\gamma_{SR}(T_j)&\le f(V|_{T_j})\\&=f(V)-f(V(P_4)).\end{aligned}$$

Now, assume that $f(u)=0$. Hence $f|_{T_j-u}$ is an SRDF on T_j-u. Therefore $\gamma_{SR}(T_j-u)\le f(V|_{T_j-u})$ and so by the assumption

$$\begin{aligned}\gamma_{SR}(T_j)&\le\gamma_{SR}(T_j-u)\\&\le f(V|_{T_j-u})\\&=f(V)-f(V(P_4)).\end{aligned}$$

On the other hand, always $f(V(P_4))\ge 5$.

So, in both cases we have

$$\begin{aligned}\gamma_{SR}(T_j)&\le f(V)-f(V(P_4))\\&\le f(V)-5\\&=\gamma_{SR}(T_{j+1})-5.\end{aligned}$$

Therefore $\gamma_{SR}(T_{j+1})=\gamma_{SR}(T_j)+5$. Since $\deg(u)\ge 2$, we get $l_{j+1}=l_j+2$ and $s_{j+1}=s_j+2$.

So, by induction we have

$$\begin{aligned}\gamma_{SR}(T_{j+1})&=\gamma_{SR}(T_j)+5\\&=\frac{6n_j-l_j-s_j}{4}+5\\&=\frac{6(n_{j+1}-4)-(l_{j+1}-2)-(s_{j+1}-2)}{4}+5\\&=\frac{6n_{j+1}-l_{j+1}-s_{j+1}}{4}.\end{aligned}$$

Now, hence the proof. □

Lemma 2: Suppose that $\gamma_{SR}(T_j)=\frac{6n_j-l_j-s_j}{4}$ and T_{j+1} is obtained with the operation O_2 of T_j, then

$$\gamma_{SR}(T_{j+1})=\frac{6n_{j+1}-l_{j+1}-s_{j+1}}{4}.$$

Proof:

Let F_3 be a tree with central vertex v and vertex $u\in V(T_j)$ is dependent to operation O_2. Suppose that f is a $\gamma_{SR}(T_{j+1})-function$.

If $f(u)\neq 0$, then $f|_{T_j}$ is an SRDF on tree T_j. Thus

$$\begin{aligned}\gamma_{SR}(T_j)&\le f(V|_{T_j})\\&=f(V)-f(V(F_3)).\end{aligned}$$

Now, let $f(u)=0$. In this case, we have

$$\begin{aligned}\gamma_{SR}(T_j)&\le\gamma_{SR}(T_j-u)\\&\le f(V|_{T_j-u})\\&=f(V)-f(V(F_3)).\end{aligned}$$

On the other hand, always $f(V(F_3))\ge 9$.

So, in both cases we have

$$\begin{aligned}\gamma_{SR}(T_j)&\le f(V)-f(V(F_3))\\&\le f(V)-9\\&=\gamma_{SR}(T_{j+1})-9.\end{aligned}$$

Also any $\gamma_{SR}(T_j)-function$ can be extended to an SRDF on tree T by assigning the weight 3 to the support vertices of tree F_3 and 0 to other vertices of tree F_3. Thus

$$\begin{aligned}\gamma_{SR}(T_{j+1})&\le\gamma_{SR}(T_j)+3|S(F_3)|\\&=\gamma_{SR}(T_j)+9.\end{aligned}$$

Therefore $\gamma_{SR}(T_{j+1})=\gamma_{SR}(T_j)+9$. Clearly $n_{j+1}=n_j+7$ and since $\deg(u)\ge 2$, we get $l_{j+1}=l_j+3$ and $s_{j+1}=s_j+3$. So, by induction we have

$$\begin{aligned}\gamma_{SR}(T_{j+1})&=\gamma_{SR}(T_j)+9\\&=\frac{6n_j-l_j-s_j}{4}+9\\&=\frac{6(n_{j+1}-7)-(l_{j+1}-3)-(s_{j+1}-3)}{4}+9\\&=\frac{6n_{j+1}-l_{j+1}-s_{j+1}}{4}.\end{aligned}$$

Now, hence the proof. □

Theorem 7: For any tree T of order $n \geq 3$ with l leaves and s support vertices, $\gamma_{SR}(T) = \dfrac{6n-l-s}{4}$ if and only if $T \in F$.

Proof:

Suppose that $\gamma_{SR}(T) = \dfrac{6n-l-s}{4}$. We proceed by an induction on the order n of a tree T.

If $diam(T) \leq 3$, then based on proof of Theorem 6, we get $T = P_4$ and thus $T \in F$.

Now, let $diam(T) \geq 4$. We root the tree T at vertex x_0.

Suppose that $P = x_0\, x_1 \ldots x_d$ isa diagonal path. Based on proof of Theorem 6, T does not have a Strong support vertexand if $T \neq F_3$, then only in following two cases are $\gamma_{SR}(T) = \dfrac{6n-l-s}{4}$ holds:

Case 1: $\deg(x_{d-2}) = 3$, $\deg(x_{d-2}) \geq 3$ and x_{d-2} is a support vertex.

In this case, we put $T' = T - T_{x_{d-2}}$. Let u be adjacent leaf to x_{d-2}. Hence $T_{x_{d-2}} \cong P_4$. To prove that T is obtained from T' with operation O_1, it is enough to show $\gamma_{SR}(T' - x_{d-3}) \geq \gamma_{SR}(T')$.

Let $T' \in F$. On contrary, suppose that $\gamma_{SR}(T' - x_{d-3}) < \gamma_{SR}(T')$. Any $\gamma_{SR}(T' - x_{d-3}) -$ *function* can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-2}, 2 to x_d and 0 to x_{d-1}, x_{d-3} and u. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T' - x_{d-3}) + 5$. Thus

$$\begin{aligned}\gamma_{SR}(T) &\leq \gamma_{SR}(T' - x_{d-3}) + 5\\ &< \gamma_{SR}(T') + 5\\ &\leq \frac{6n'-l'-s'}{4} + 5\\ &= \frac{6(n-4)-(l-2)-(s-2)}{4} + 5\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

Therefore $\gamma_{SR}(T) < \dfrac{6n-l-s}{4}$ which is a contradiction. So, $\gamma_{SR}(T' - x_{d-3}) \geq \gamma_{SR}(T')$.

Now, let $T' \notin F$. In this case, $\gamma_{SR}(T') < \dfrac{6n'-l'-s'}{4}$. Any $\gamma_{SR}(T') -$ *function* can be extended to an SRDF by assigning the weight 3 to v, 0 to u and x_{d-1} and 2 to x_d. Thus

$$\begin{aligned}\gamma_{SR}(T) &\leq \gamma_{SR}(T') + 5\\ &< \frac{6n'-l'-s'}{4} + 5\\ &= \frac{6(n-4)-(l-2)-(s-2)}{4} + 5\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

Therefore $\gamma_{SR}(T) < \dfrac{6n-l-s}{4}$ which is a contradiction.

Hence $T' \in F$. Thus T is obtained from T' with operation O_1.

Case 2: $\deg(x_{d-2}) = 4$, $\deg(x_{d-2}) \geq 3$ and x_{d-2} has exactly two support children u and v other than x_{d-1}.

We put $T' = T - T_{x_{d-2}}$. In this case, $T_{x_{d-2}} \cong F_3$. Any $\gamma_{SR}(T') -$ *function* can be extended to an SRDF by assigning the weight 3 to x_{d-1}, u and v and 0 to all their neighboring vertices. Thus $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 9$.

If $T' \notin F$, then $\gamma_{SR}(T') < \dfrac{6n'-l'-s'}{4}$.

Therefore

$$\begin{aligned}\gamma_{SR}(T) &\leq \gamma_{SR}(T') + 9\\ &< \frac{6n'-l'-s'}{4} + 9\\ &= \frac{6(n-7)-(l-3)-(s-3)}{4} + 9\\ &= \frac{6n-l-s}{4}.\end{aligned}$$

So, $\gamma_{SR}(T) < \dfrac{6n-l-s}{4}$ which is a contradiction. Hence $T' \in F$.

To prove that T is obtained from T' with operation O_2, it is enough to show that

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$\gamma_{SR}\left(T'-x_{d-3}\right)\geq\gamma_{SR}\left(T'\right)$. Any $\gamma_{SR}\left(T'\right)-$ *function* can be extended to an SRDF by assigning the weight 3 to x_{d-2}, 0 to x_{d-3} and vertices $S\left(T_{x_{d-2}}\right)$ and 2 to $L\left(T_{x_{d-2}}\right)$. Thus $\gamma_{SR}\left(T\right)\leq\gamma_{SR}\left(T'-x_{d-3}\right)+9$. Therefore

$$\begin{aligned}\gamma_{SR}\left(T\right)&\leq\gamma_{SR}\left(T'-x_{d-3}\right)+9\\&<\gamma_{SR}\left(T'\right)+9\\&\leq\frac{6n'-l'-s'}{4}+9\\&=\frac{6\left(n-7\right)-\left(l-3\right)-\left(s-3\right)}{4}+9\\&=\frac{6n-l-s}{4}.\end{aligned}$$

Thus $\gamma_{SR}\left(T\right)<\frac{6n-l-s}{4}$ which is a contradiction.

So $\gamma_{SR}\left(T'-x_{d-3}\right)\geq\gamma_{SR}\left(T'\right)$. Therefore T is obtained from T' with operation O_2. Hence $T\in F$.

Hence in both cases $T\in F$.

Conversely, let $T\in F$. We apply induction on the number of operations performed to construct a tree T.

If $T=P_4$ or $T=F_3$, then clearly $\gamma_{SR}\left(T\right)=\frac{6n-l-s}{4}$.

Now, let $T\neq P_4$ and $T\neq F_3$. Based on the structure of F, let T be obtained of $T'\in F$ with operations O_1 and O_2. Under the hypothesis we have $\gamma_{SR}\left(T'\right)=\frac{6n'-l'-s'}{4}$ where n', l' and s' denote order , number of leaves and number of vertices of tree T', respectively.

If T is obtained from T' with operation O_1, then based on Lemma 1 we have $\gamma_{SR}\left(T\right)=\frac{6n-l-s}{4}$.

Also, if T is obtained from T' with operation O_2, then from the Lemma 2 it follows that $\gamma_{SR}\left(T\right)=\frac{6n-l-s}{4}$.

Hence the proof. □

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