wgrα-I-Homeomorphism in Ideal Topological Spaces

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Abstract: In this paper, the concepts of wgrα-I-closed maps, wgrα-I-homeomorphism, wgrα-I-connectedness and wgrα-I-compactness are introduced and some their properties in ideal topological spaces are investigated.

Keywords: *wgrα-I-homeomorphism, wgrα*-I-homeomorphism, wgrα-I-closed and wgrα-I-open maps, wgrα-I-connectedness and wgrα-I-compactness.*
Subject Classification: 54A05, 54C05.

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I. Introduction

The concept of ideal in topological space was first introduced by Kuratowski[10] and Vaidyanathaswamy [13]. They also have defined local function in ideal topological space. Further Hamlett and Jankovic [8] studied the properties of ideal topological spaces. Using the local function, they defined a kuratowski closure operator in new topological space .Compactness [5, 11, 12], connectedness have been generalized via topological ideals in the recent years. In this paper we introduce and study some of the properties of wgrα-I-closed and wgrα-I-open maps. Further, we introduce two new homeomorphisms namely wgrα-I-homeomorphism, wgrα*-I-homeomorphism. Also, the concept of wgrα-I-connectedness and wgrα-I-compactness are introduced in ideal topological spaces.

II. Preliminaries

Definition:2.1

A subset A of a space (X,τ) is called

(i) wgrα-closed[9] if cl(int(A))⊆U whenever A⊆U and U is regular α-open.

(ii) α-open[4] if A⊆int(cl(int(A)).

Definition: 2.2

A subset A of (X,τ,I) is said to be

(i)wgrα-I-closed[6]ifcl*(int(A))⊆U whenever A⊆U and U is regular α-open.

(ii) α-I-closed [1,2] if $cl(int^*(cl(A)))\subseteq A$.

Definition: 2.3

A function f:(X,τ,I) →(Y,σ,J) is said to be

(i) wgrα-I-continuous[7] if $f^{-1}(V)$ is wgrα-I-closed in X for every closed set V of Y.

(ii) wgrα-I-irresolute[7] if $f^{-1}(V)$ is wgrα-I-closed in X for every wgrα-I-closed set V of Y.

(iii) strongly wgrα-I-continuous[7] if $f^{-1}(V)$ is open in X for every wgrα-I-open set V of Y.

Definition:2.4[7]

For a function f: (X,τ,I) →(Y,σ) is called contra wgrα-I-continuous if $f^{-1}(V)$ is wgrα-I-open in (X,τ,I) for every closed set V of (Y,σ).

III. Wgrα-I-closed maps

Definition: 3.1

A map f: (X,τ)→(Y,σ,J) is called wgrα-I-closed if f(V) is a wgrα-I-closed set of (Y,σ,J) for each closed set V of (X,τ) .

Definition :3.2

A map f: (X,τ,I)→(Y,σ,J) is called prewgrα-I-closed if f(V) is a wgrα-I-closed set of (Y,σ,J) for every wgrα-I-closed set V of (X,τ,I) .

Theorem :3.3

(i) Every closed map is wgrα-I-closed map.

(ii)Every α-I-closed map is wgrα-I-closed map.

(iii)Every wgrα-closed map is wgrα-I-closed map.

(iv)Every prewgrα-I-closed map is wgrα-I-closed map.

Proof Straightforward.

Remark :3.4

The converse of the above theorem need not be true as seen in the following examples.

Example :3.5

Let X = Y = {a,b,c,d},τ = {φ,X, {a}, {a,b}, {a,b,d}},σ ={φ,Y,{a},{a,c,d}},I = {φ,{a}} and f be an identity map. Thus, f is wgrα-I-closed map, but not closed map.

Example :3.6

Let X=Y={a,b,c},τ = {φ,{a},X}, σ={X, φ, {b}, {c},{b,c}}, I={φ,{c}} and f be an identity map. Here f is wgrα-I-closed map, but not α-I-closed map.

Example :3.7

Let X=Y={a,b,c,d},τ = {φ,X,{a}, {c}, {a,c}, {a,c,d}}, I={φ,{b}}, σ={φ,Y,{a},{b},{a,b}, {b,c}, {a,b,c}} and f be an identity map. Here f is wgrα-I-closed map, but not wgrα-I-closed map.

Example :3.8

Let X =Y= {a,b,c,d},τ = {φ,X,{a}, {c,d}, {a,c,d}}, I={φ,{a}}, σ={φ,Y,{a},{a,c,d}} and f be a map defined by f(a) ={c}, f(b) = {d} , f(c) = {a} and f(d)={b}. Here f is wgrα-I-closed map, but not prewgrα-I-closed map.

Theorem :3.9

If f: (X,τ,I)→(Y,σ,J) is wgrα-I-closed and A is wgrα-I-closed set of X. then f|A : (A,τ|A,I) →(Y,σ,J) is wgrα-I-closed.

Proof

Let F be a closed set of A. Since F is closed in X. (f|A)(F)=f(F) is wgrα-I-closed in Y. Hence f|A is a wgrα-I-closed map.

Theorem :3.10

A map f: (X,τ,I)→(Y,σ,J) is wgrα-I-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$, there exists an wgrα-I-open set V of Y containing S and $f^{-1}(V)\subset U$.

Proof

Let S be a subset of Y and U be open set of X such that $f^{-1}(s)\subset U$, then V= Y–f(X–U) is a wgrα-I-open set containing S such that $f^{-1}(V)\subset U$.

Conversely, suppose that F is a closed set of X. Then $f^{-1}(Y–f(F))\subset X–F$ and X–F is open. By hypothesis, there is a wgrα-I-open set V of Y such that $Y–f(F)\subset V$ and $f^{-1}(V)\subset X–F$. Therefore, $F\subset X–f^{-1}(V)$. Hence $Y–V\subset f(F)\subset f(X–f^{-1}(V))\subset Y–V$, which implies that f(F)=Y–V. Since Y–V is wgrα-I-closed, f(F) is wgrα-I-closed and thus f is wgrα-I-closed map.

Theorem :3.11

If f : X→Y is a bijection mapping , then the following statements are equivalent.

(i) f is a wgrα-I-open.

(ii) f is a wgrα-I-closed.

(iii) f^{-1} : Y→X is wgrα-I-continuous.

Proof

(i)$\Rightarrow$(ii) Let U be closed in X and f be a wgrα-I-open map. Then X–U is open in X. By hypothesis, we get f(X–U) is a wgrα-I-open in Y. Hence f(U) is wgrα-I-closed in Y.

(ii)$\Rightarrow$(iii)

Let U be closed in X. By (ii), f(U) is wgrα-I-closed in Y. Hence f^{-1} is wgrα-I-continuous.

(iii)$\Rightarrow$(i)

Let U be open in X. As $(f^{-1})^{-1}(U)=f(U)$, f is wgrα-I-open map.

Theorem :3.12

For any bijection f: (X,τ,I)⟶(Y,σ,J) the following statements are equivalent.

(i) Its inverse map f^{-1} :(Y,σ,J)⟶(X,τ,I) is prewgrα-I-open map.

(ii) f is a prewgrα-I-open map.

(iii) f is a prewgrα-I-closed map.

Proof

(i) $\Rightarrow$(ii)Let V be wgrα-I-open in (X, τ, I).By hypothesis, $(f^{-1})^{-1}(V)$ = f(V) is wgrα-I-open in (Y, σ, J). Hence (ii) holds.

(ii) $\Rightarrow$(iii)Let V be wgrα-I-closed in (X, τ, I),then X-V is wgrα-I-open . f(X-V)=Y-f(V) is wgrα-I-open in (Y, σ, J), since f is a prewgrα-I-open map. That is f(V) is wgrα-I-closed in Y and so f is prewgrα-I-closed map.

(iii) $\Rightarrow$(i)Let V be wgrα-I-closed in (X,τ, I). By (iii), f(V) is wgrα-I-closed in (Y, σ, J). But f(V)= $(f^{-1})^{-1}(V)$. Thus (i) holds.

Theorem :3.13

If a map f: (X,τ,I)→(Y,σ,J) is closed and a map g: (Y,σ,J)⟶(Z,η,K) is wgrα-I-closed, then $g\circ f$: (X,τ,I)→(Z,η,K) is a wgrα-I-closed map.

Proof Let V be a closed set in X, then f(V) is open and $(g\circ f)(V)$ = g(f(V)) is wgrα-I-closed. Hence $g\circ f$ is wgrα-I-closed.

Theorem :3.14

Let f: (X,τ,I)→(Y,σ,J), g: (Y,σ,J)→(Z,η,K) be two mappings and let $g\circ f$: (X,τ,I)→(Z,η,K) be wgrα-I-closed map. Then

(i)If f is continuous and surjective, then g is wgrα-I-closed

(ii) If g is wgrα-I-irresolute and injective, then f is wgrα-I-closed.

(iii) If g is strongly wgrα-I-continuous and injective, then f is closed.

Proof

(i)If f is continuous, then for any closed set A of Y, $f^{-1}(A)$ is closed in X. As, $g \circ f$ is wgrα-I-closed, g(A) is wgrα-I-closed in Z and g is wgrα-I-closed map.

(ii)Let A be closed in (X,τ,J). Then $(g \circ f)(A)$ is wgrα-I-closed in Z and $g^{-1}(g \circ f)(A)=f(A)$ is wgrα-I-closed in Y. Hence f is wgrα-I-closed.

(iii)Let A be closed in X, then $(g \circ f)(A)$ is wgrα-I-closed in Z . g is strongly wgrα-I-continuous implies $g^{-1}(g \circ f)(A)= f(A)$ is closed in Y and f is a closed map.

IV. wgrα-I-homeomorphism

Definition :4.1

A bijection function f: (X,τ,I)→(Y,σ,J) is called

(i) wgrα-I-homeomorphism if both f and f^{-1} are wgrα-I-continuous.

(ii) wgrα*-I-homeomorphism if both f and f^{-1} are wgrα-I-irresolute.

The family of all wgrα-I-homeomorphism (resp wgrα*-I-homeomorphism) from (X,τ,I) onto itself is denoted by wgrα-I-h(X,τ,I)(resp.wgrα*-I-h(X,τ,I)).

Theorem :4.2

Every homeomorphism is a wgrα-I-homeomorphism.

Proof

Let f: (X,τ,I)→(Y,σ,J) be a homeomorphism. Then f and f^{-1} are continuous and f is bijection. As every I-continuous function is wgrα-I-continuous, we have f and f^{-1} are wgrα-I-continuous. Therefore, f is wgrα-I-homeomorphism.

Remark :4.3

The converse of the above theorem need not be true as seen from the following example.

Example :4.4

Let X={a, b, c} =Y,τ={φ,X,{a},{c}, {a,c,d}}, I={φ,{a}},σ ={φ,Y, {a}, {b}, {a,b}, {b,c}, {a,b,c}} . The mapping f : (X,τ,I)→(Y,σ,J) defined as f(a)= c, f(b)=a, f(c)=d and f(d)=b. Therefore f is wgrα-I -homeomorphism, but it is not homeomorphism.

Theorem :4.5

Let f : (X,τ,I)→(Y,σ,J) be a bijective and wgrα-I-continuous. Then the following statements are equivalent.

(i) f is wgrα-I-open map.

(ii) f is wgrα-I-homeomorphism.

(iii) f is an wgrα-I-closed map.

Proof

(i)$\Rightarrow$(ii)

Suppose f is bijective, wgrα-I-continuous and wgrα-I-open. Then f is wgrα-I-homeomorphism.

(ii)$\Rightarrow$(iii)

Let F be a closed set of (X,τ,I).Then $(f^{-1})^{-1}(F)=f(F)$ is wgrα-I-closed set in Y, since f and f^{-1} are wgrα-I-continuous. So f is wgrα-I-closed map.

(iii)$\Rightarrow$(i)

Proof obvious.

Theorem :4.6

For any space wgrα*-I-h(X,τ,I)wgrα-I-h(X,τ,I).

Proof

The proof follows from the fact that every wgrα-I-irresolute function is wgrα-I-continuous and every prewgrα-I-open map is wgrα-I-open.

Remark :4.7

The composition of two wgrα-I-homeomorphism need not be a wgrα-I-homeomorphism as seen from the following example.

Example :4.8

Let X={a,b,c} =Y,τ={φ,X,{a}, {b},{a,b} }, I={φ,{a}},σ={φ,Y,{a},{a,c}} and J= {φ, {a}}. Let f be the map defined by f(a)=b , f(b)=a and f(c)=c and g be the identity map. Therefore f and g are wgrα-I-homeomorphism, but $g \circ f$ is not wgrα-I-homeomorphism.

Theorem:4.9

Let f : (X,τ,I)→(Y,σ,J) and g : (Y,σ,J) → (Z,η,K) are wgrα*-I-homeomorphism, then their composition $g \circ f$: (X,τ,I)→(Z,η,K) is also wgrα*-I-homeomorphism.

Proof

Let U be a wgrα-I-open set in (Z,η,K).Since g is wgrα-I-irresolute,$g^{-1}(U)$ is wgrα-I-open in (Y,σ,J).Since f is wgrα-I-irresolute, $f^{-1}(g^{-1}(U))=(g \circ f)^{-1}(U)$ is wgrα-I-open set in (X,τ,I).Therefore $g \circ f$ is wgrα-I-irresolute. Also, for a wgrα-I-open set G in (X,τ,I), we have $(g \circ f)(G)= g(f(G)) = g(W)$, where W=f(G). By hypothesis f(G) is wgrα-I-open in (Y,σ,J) and so again by hypothesis g(f(G)) is wgrα-I-open in (Z,η,K). That is, $(g \circ f)$ (G) is a wgrα-I-open set in (Z,η,K) and therefore,$g \circ f$ is wgrα-I-irresolute. Also, $g \circ f$ is a bijection. Hence $g \circ f$ is wgrα*-I-homeomorphism.

Theorem :4.10

Let f: (X,τ,I)→(Y,σ,J) is a wgrα*-I-homeomorphism, then it induces an isomorphism from the group wgrα*-I-h(X,τ,I) onto the group wgrα*-I-h(Y,σ,J).

Proof

Using the map f, we define a map ψ_f: wgrα*-I-h(X,τ,I)→(Y,σ,J) by $\psi_f(h)= f \circ h \circ f^{-1}$ for every $h \in$ wgrα*-

I-h(X,τ,I). Then ψ_f is a bijection. Further, for all $h_1, h_2 \in$ wgrα*-I-h(X,τ,I), $\psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1}$

$$= (f \circ h_1 \circ f^{-1})(f \circ h_2 \circ f^{-1})$$
$$= \psi_f(h_1) \circ \psi_f(h_2).$$

Therefore, ψ_f is a homeomorphism and so it is an isomorphism induced by f.

Theorem :4.11

The set wgrα*-I-h(X,τ,I) is group under the composition of maps.

Proof

Define a binary relation *: wgrα*-I-h(X,τ,I)→wgrα*-I-h(X,τ,I) f*g=g∘f for all f, g∈wgrα*-I-h(X,τ,I) and ∘ is the usual operation of composition of maps Then by theorem, g∘f∈wgrα*-I-h(X,τ,I). We know that the composition of maps is associative and the identity map I: (X,τ,I)→(X,τ,I) belonging to wgrα*-I-h(X,τ,I) serves as the identity element. For any f∈wgrα*-I-h(X,τ,I), $f \circ f^{-1} = f^{-1} \circ f = I$. Hence inverse exists for each element of wgrα*-I-h(X,τ,I).Then wgrα*-I-h(X,τ,I), form a group under the operation, composition of maps.

V. Wgrα-I-connectedness

Definition:5.1

An ideal topological space (X,τ,I) is said to be wgrα-I-connected if X cannot be written as the disjoint union of two non-empty wgrα-I-open sets. If X is not wgrα-I-connected it is said to be wgrα-I-disconnected.

Theorem:5.2

Let (X,τ,I) be an ideal topological space. If X is wgrα-I-connected, then X cannot be written as the union of two disjoint non-empty wgrα-I-closed sets.

Proof

Suppose not,that is X=A∪B, where A and B are wgrα-I-connected sets, A≠φ, B≠φ, and A∩B=φ. Then $A=B^C$ and $B=A^C$. Since A and B are wgrα-I-closed sets, which implies that A and B are wgrα-I-open sets. Therefore X is not wgrα-I-connected, which is a contradiction. Hence the proof.

Theorem:5.3

For an ideal topological space (X,τ,I), the following are equivalent.

(i)X is wgrα-I-connected.

(ii) X and φ are the only subsets of X which are both wgrα-I-open and wgrα-I-closed.

Proof Obvious.

Theorem:5.4

Let f: (X,τ,I)→(Y,σ) be a function. If X is wgrα-I-connected and f is wgrα-I-irresolute, surjective, then Y is wgrα-I-connected.

Proof

Suppose that Y is not wgrα-I-connected. Let Y=A∪B, where A and B are disjoint non-empty wgrα-I-open sets in Y. Since f is wgrα-I-irresolute and onto, $f^{-1}(Y)=f^{-1}(A\cup B)$ which implies $X= f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A)\cap f^{-1}(B) = f^{-1}(A\cap B)=f^{-1}(\varphi)=\varphi$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty wgrα-I-open sets in X. This contradicts to the fact that X is wgrα-I-connected. Hence Y is wgrα-I-connected.

Theorem:5.5

Let f: (X,τ,I)→(Y,σ) be a function. If X is wgrα-I-connected and f is wgrα-I-continuous, surjective, then Y is connected.

Proof

Suppose that Y is not connected. Let Y=A∪B, where A and B are disjoint non-empty open sets in Y.Since f is wgrα-I-continuous surjective, therefore $X=f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty wgrα-I-open sets in X. This contradicts the fact that X is wgrα-I-connected. Hence Y is connected.

Theorem:5.6

Every wgrα-I-connected space is connected

Proof

Let X be wgrα-I-connected. Suppose X is not connected, then there exists a proper non-empty subset B of X which is both open and closed in X. Since every closed set is wgrα-I-closed, B is a proper non-empty subset of X which is both wgrα-I-open and wgrα-I-closed in X . Then by theorem , X is not wgrα-I-connected. This proves the theorem.

Remark :5.7

The converse of the above theorem need not be true, which has seen from the following example.

Example:5.8

Let X={a,b,c,d},τ = {φ, X,{a}, {b}, {a,b}, {b,c},{a,b,c}} and I = {φ,{a}}.In this ideal space {c,d} is connected, but not wgrα-I-connected.

Theorem:5.9

If X is connected and f is continuous surjective, then Y is wgrα-I-connected.

Proof

Suppose that Y is not wgrα-I-connected. Let Y=A∪B, where A and B are disjoint non-empty wgrα-I-open sets in Y, so they are open. Since f is continuous surjective, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint open sets in X and $X=f^{-1}(A) \cup f^{-1}(B)$. This contradicts the fact that X is connected, therefore Y is connected.

VI. Wgrα-I-compactness

Definition:6.1

A collection $\{A_\alpha : \alpha\in\nabla\}$ of wgrα-I-open sets in a topological space X is called a wgrα-I-open cover of a subset B of X if $B\subseteq\cup\{A_\alpha : \alpha\in\nabla\}$ holds.

Definition:6.2

An ideal topological space (X,τ,I) is said to be wgrα-I-compact if every wgrα-I-open cover of X has a finite subcover.

Definition:6.3

A subset B of an ideal topological space (X,τ,I) is said to be wgrα-I- compact relative to X if for every collection $\{A_\alpha : \alpha \in\nabla\}$ of wgrα-I-open subsets of X such that $B\subseteq\cup\{A_\alpha : \alpha\in \nabla\}$ there exists a finite subset ∇_0 of ∇ such that $B\subseteq\{A_\alpha : \alpha \in\nabla_0\}$.

Theorem:6.4

(i)A wgrα-I-continuous image of a wgrα-I-compact space is compact.

(ii) If a map f:(X,τ,I)→(Y,σ) is rps-I-irresolute and a sunset B of X, then f(B) is wgrα-I-compact relative to Y.

Proof

(i)Let f: (X,τ,I)→(Y,σ) be a wgrα-I-continuous map from a wgrα-I-compact space X onto a topological space Y.Let $\{V_\alpha : \alpha\in\nabla\}$ be an open cover of T. Then $\{f^{-1}(V_\alpha) : \alpha\in\nabla\}$ is wgrα-I-open cover of X. Since X is wgrα-I-compact, it has a finite subcover, say$\{f^{-1}(V_1), f^{-1}(V_2),\ldots, f^{-1}(V_n)\}$. Since f is onto,$\{V_1,V_2,\ldots,V_n\}$ is a cover of Y and so Y is compact.

(ii) Let $\{V_\alpha : \alpha\in\nabla\}$ be any collection of wgrα-I-open subsets of Y such that $f(B)\subseteq\cup\{V_\alpha : \alpha\in\nabla\}$, then $B\subseteq\cup \{f^{-1}(V_\alpha) : \alpha \in\nabla\}$holds. By hypothesis, there exists a finite subset ∇_0 of ∇ such that $B\subseteq\cup \{f^{-1}(V_\alpha) : \alpha\in\nabla_0\}$.Therefore, we have $f(B)\subseteq\cup\{V_\alpha : \alpha\in\nabla_0\}$,which shows that f(B) is wgrα-I-compact relative to Y.

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