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QUAL HUE COLORING FOR GRAPHS UNION

[1]**SRI DEVI M J**
Department of Mathematics
Government Science College (Autonomous)
Hassan -573 201, India.
Email Id: ***thanusri.j@gmail.com***

[2]**JAGADEESHA .R.D.**
Department of Mathematics
Government Science College (Autonomous)
Hassan -573 201, India.
Email Id: jagdeesh25docs@gmail.com

S. Vijay (Corresponding Author)
Department of Mathematics
Government Science College (Autonomous)
Hassan -573 201, India.
Email Id: vijayshivanna82@gmail.com

Abstract—Let G be a simple, finite, and undirected graph. If there is at most a one-size difference in the sizes of the color classes, then a graph of G has an equitable vertex coloring. The least k that makes a graph G equitably k-colorable is its equitable chromatic number, shown by$\boldsymbol{\chi_{=}(G)}$,\. We will talk about a fair coloring scheme for the union of two graphs.

Keywords-undircted graph, vertex coloring, finite

I. Introduction

All graphs considered in this paper are finite, undirected and without loops and multiple edges. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Any definitions not covered in this study can be found at [2, 3]. Euler's work on the Königsberg bridge issues is where graph theory first emerged [1]. It may be used for many different things, such determining communities inside networks, figuring out the shortest path, examining chemical structures, and more. Since the four color problem's inception in 1852, graph coloring has grown to become one of graph theory's most fascinating subfields [7]. Graph coloring, in particular, is essential to computer science and discrete mathematics. The applications of these coloring challenges have been the focus of several research publications over the previous few decades. A proper $k-$ coloring of a graph G is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ define in such a way that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The vertices of the samecolor form a color class. The chromatic number $\chi(G)$ of a graph G, is the smallest integer k such that G has a proper $k-$ coloring. An edge coloring assigns a color to each so that no two adjacent edges share the same color. In this work, we concentrate on equitable coloring, a common application of graph coloring [5]. Meyer [4] established the idea of fair colorability for the first time. Tucker's program, which had vertices representing garbage collection routes connected when comparable routes shouldn't be performed on the same day, served as his inspiration. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$such that each V_i is an independent set and the condition $\|V_i| - |V_j\| \leq 1$ holds for every pair (i, j), then G is said to be equitably $k -$ colorable. The smallest integer k for which G is equitably $k-$ colorable is known as the equitable chromatic number of [13–16] G and is denoted by $\chi_{=}(G)$. Since equitable coloring is a proper coloring with additional condition, $\chi(G) \leq \chi_{=}(G)$ for any graph G. It is interesting to note that if a graph G is equitably $k -$ colorable, it does not imply that it is equitably $k + 1-$ colorable. A counter example is the complete bipartite graph $K_{3,3}$ which can be equitably colored with two colors, but not with three. The equitable chromatic threshold of $G$is $\chi^*_{=}(G) = min\{t : G\ is\ equitably\ k - colorable\ for\ all\ k \geq t\}$. In 1964, Erdos [8] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k+1)-$ coloring, or equivalently is $\chi^*_{=}(G) \leq \Delta(G) + 1$. Hajnal and Szemeredi proved this conjecture in 1970 [9]. Recently, a polynomial technique for such a coloring was described by Kierstead and Kostochka [10], along with a brief demonstration of the theorem. Meyer [4] proposed the following hypothesis in 1973: Conjecture on Equitable Coloring [4]. For any connected graph G, other than a complete graph or an odd cycle, $\chi_{=}(G) \leq \Delta(G)$.For any graphs with six vertices or less, this conjecture has been confirmed. The Equitable Coloring Conjecture holds for all bipartite graphs, as demonstrated by Lih and Wu [16]. Wang and Zhang [19] examined r-partite graphs, a more general type of graphs. Meyer's hypothesis holds for entire graphs in this class, as they demonstrated. Furthermore, the hypothesis was verified for planar graphs with a maximum degree of at least 13 [18] and outerplanar graphs [17]. We also have a more robust hypothesis: Equitable $\Delta-$ Coloring conjecture [13], If G is a connected graph of degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then G is equitably $\Delta-$ Colorable. The Equitable Δ Coloring Conjecture holds for some

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classes of graphs, e.g., bipartite graphs [16], outerplanar graphs with $\Delta \geq 3$ [17] and planar graphs with $\Delta \geq 13$ [18]. The detailed survey of this type of coloring is found in Lih [6]. In the present paper, we study on equitable coloring for union of graphs.

II. PRELIMINARIES

We would want to discuss some early findings about equitable coloring before moving on to the major findings.

Definition 2.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. $G_1 \cup G_2$ denotes the Union of two graphs G_1 and G_2 has the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. When G_1 and G_2 are disjoint $G_1 \cup G_2$ is denoted by $G_1 + G_2$.

Lemma 2.2. [20] If two graphs G and H with disjoint vertex sets are both equitably $k-$ colorable, then $G + H$ is also equitably $k-$ colorable.

Lemma 2.3. [20] $mK_{n,n}$ is equitably $k-$ colorable for any $m \geq 2, n \geq 2$ and $k \geq 2$.

Lemma 2.4. [20] Let G be a graph and suppose that $|V(G)|$ is not divisible by a positive integer $n \geq 3$. If G is equitably $n-$ colorable, then $G + K_{n,n}$ is also equitably $n-$ colorable.

Lemma 2.5. [20] Let G be a graph and suppose that $|V(G)|$ is divisible by a positive integer $n \geq 3$. If there exists a proper $n-$ coloring of G such that the sizes of color classes in nondecreasing order are $\frac{|V(G)|}{n} - 1, \frac{|V(G)|}{n} - 2 \ . \ . \ . \ , \frac{|V(G)|}{n}, \frac{|V(G)|}{n} + 1$, then $G + K_{n,n}$ is equitably $n-$ $colorable.$

Lemma 2.6. [20] Let $n \geq 2$ be a positive integer and let G be a graph with $\Delta(G) \leq n-1$. Then $G + K_{n,n}$ is equitably $n-$ colorable if and only if n is even, or G is different from mK_n for all $m \geq 1$.

Lemma 2.7. [20] Let G be a graph with $\Delta(G) \geq \chi(G)$. If G is equtably Δ(G)− colorable, then at least one of the following statements holds.

1. $\Delta(G)$ is even.

2. No components or at least two components of G are isomorphic to $K_{\Delta(G),\Delta(G)}$.

3. Only one component of G is isomorphic to $K_{\Delta(G),\Delta(G)}$ and $\alpha(G - K_{\Delta(G),\Delta(G)}) > \frac{|V(G - K_{\Delta(G),\Delta(G)})|}{\Delta} > 0.$

III. FAIR COLORING FOR THE INTERSECTION OF TWO GRAPHS

The generalized formula for the equitable chromatic number for the union of any two graphs was found in this section.

Theorem 3.1. Let G_1 and G_2 be two graphs. Let $k = max(\chi^*_=(G_1), \chi^*_=(G_2))$. Then $G_1 \cup G_2$ is equitably $k-$ colorable.

Proof. Let G_1 and G_2 be two graphs with n_1 and n_2 vertices respectively. Let max $\left(\chi^*_=(G_1), \chi^*_=(G_2)\right) = k$. With out loss of generality, let $\chi^*_=(G_1) = k$. Case 1: Let $n_2 \geq k$. $\chi^*_=(G_2) \leq k \leq n_2$. Since G_2 is equitably k colorable. Let $\varphi_1 = \{V_1, V_2, \ldots, V_k\}$ be an equitable color partition of G_1 such that $|V_i| \leq |V_{i+1}|$ and $i = 1,2,\ldots,k-1$. Let $\varphi_2 = \{W_1, W_2, \ldots, W_k\}$ be an equitable color partition of G_2 such that $|W_i| \leq |W_{i+1}|$ and $i = 1,2,\ldots,k-1$. Assume that the first t color classes in φ_1 has l elements and the remaining $k-t$ color classes has $l+1$ elements. Similarly the first s color classes in φ_2 has m elements and the remaining $k-s$ has color classes has $m+1$ elements. Here $0 \leq s, t \leq k$. When either s or $t \in \{0, k\}$, $\{V_1 \cup W_1, V_2 \cup W_2, \ . \ . \ . \ , V_k \cup W_k\}$ is an equitable coloring of $G_1 \cup G_2$. In this case $\chi_=(G_1 \cup G_2) \leq k$. Let $s, t \notin \{0, k\}$. Subcase 1: Let $s \leq k-t$. Consider the partition $\varphi_3 = \{V_1 \cup W_k, V_2 \cup W_{k-1}, \ldots, V_t \cup W_{k-(t-1)}, V_{t+1} \cup W_{k-t}, \ldots, V_k \cup W_1\}$. In this partition $s+t$ classes have $l+m+1$ elements and the remaining classes contain $l+m+2$ elements. Hence φ_3 is an equitable coloring of $G_1 \cup G_2$. Subcase 2: Let $t \geq k-s$. Clearly $s \geq k-t$. In this case φ_3 becomes an equitable coloring of $G_1 \cup G_2$ where each color class contains either $l+m$ elements or $l+m+1$. Hence $\chi_=(G_1 \cup G_2) \leq k = max(\chi^*_=(G_1), \chi^*_=(G_2))$. Case 2: Let $n_2 \leq k$. Since G_2 is n_2 equitably colorable, we can always obtain an equitable color partition φ_2 for G_2 with n_2 color class when each color class contains single vertex. Let $\varphi_2 = \{\{W_1\}, \{W_2\}, \ldots, \{W_{n_2}\}\}$. Clearly $\{V_1 \cup \{W_1\}, V_2 \cup \{W_2\}, \ldots, V_{n_2} \cup \{W_{n_2}\}, V_{n_2+1}, \ldots, V_k\}$ is an equitable class partition for $G_1 \cup G_2$. Hence $\chi_=(G_1 \cup G_2) \leq k$.

Note1: The upper bound for $\chi_=(G_1 \cup G_2)$ given in the above theorem is attainable.

For example1: $\chi_=(K_{1,3} \cup K_{2,7}) \leq max(\chi^*_=(K_{1,3}), \chi^*_=(K_{2,7}))$. $\chi_=(K_{1,3} \cup K_{2,7}) = 3, \chi^*_=(K_{1,3}) = 3, \chi^*_=(K_{2,7}) = 4$Therefore 3< 4. The equitable chromatic number of $(K_{1,3} \cup K_{2,7})$ is given in the following figure.

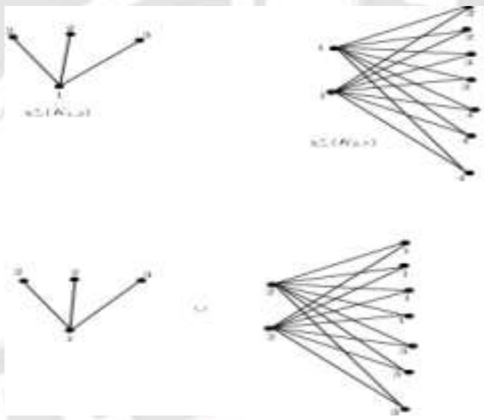

Figure 1: $\chi_=(K_{1,3} \cup K_{2,7}) = 3.$

Note2: The equality condition attain for the above theorem.

Corollary 1. $\chi_=(G_1 \cup G_2) \leq max(\chi^*_=(G_1), \chi^*_=(G_2))$.

Proof. In view of the above theorem, if $k = max(\chi^*_=(G_1), \chi^*_=(G_2))$, then $G_1 \cup G_2$ is equitably k colorable. Hence $\chi_=(G_1 \cup G_2) \leq k = max(\chi^*_=(G_1), \chi^*_=(G_2))$.

Theorem 3.2. If $G_1, G_2, \dots, G_n$ are l disjoint graphs then $\chi_{=}(\cup G_i) \leq max\ \{\chi^*_{=}(G_1), \chi^*_{=}(G_2), \dots, \chi^*_{=}(G_l)\}$.

Proof. We prove the theorem by the method of induction on n. The theorem is true for $n = 2$ in view of the above theorem. Assume that the theorem is true for $n < k$. We prove the theorem for $n = k$.

$$\chi_{=}\left(\bigcup_{i=1}^{k} G_i\right) = \chi_{=}\left(\bigcup_{i=1}^{k-1} G_i \cup G_k\right)$$
$$\leq \max(\chi^*_{=}\left(\bigcup_{i=1}^{k-1} G_i\right), \chi^*_{=}(G_k))$$
$$\leq \max(\max(\chi^*_{=}(G_1), \chi^*_{=}(G_2), \dots, \chi^*_{=}(G_{k-1})), \chi^*_{=}(G_k))$$
$$\leq \max(\chi^*_{=}(G_1), \chi^*_{=}(G_2), \dots \chi^*_{=}(G_{k-1}), \chi^*_{=}(G_k)).$$

Corollary 2.$\chi_{=}(P_m \cup P_n) = 2$.

Proof. We know that $\chi_{=}(P_m) = \chi_{=}(P_n) = 2$, $\chi^*_{=}(P_m) = \chi*=Pn=2$ for all m,n. So $2=\chi=(Pm)\leq\chi=Pm\cup Pn\leq\max 2,2$. Hence $\chi_{=}(P_m \cup P_n) = 2$.

Corollary 3.$\chi_{=}(P_n \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Proof. It is easy to verify that $\chi_{=}(P_n) = \chi^*_{=}(P_n) = 2$, $\chi_{=}(C_m) = \chi^*_{=}(C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$. By the above theorem, $\chi_{=}(P_n \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Corollary 4.$\chi_{=}(C_n \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Proof. Similarly from the above corollary, $\chi_{=}(C_n \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

The equitable chromatic number of $(C_5 \cup C_6)$ is given in the following figure.

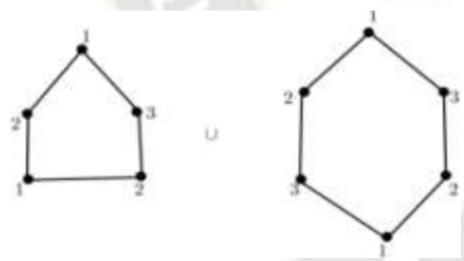

Figure 2: $\chi_{=}(C_5 \cup C_6) = 3$

Corollary 5.$\chi_{=}(K_n \cup P_n) = n$.

Proof. It is easy to verify that $\chi_{=}(P_n) = \chi^*_{=}(P_n) = 2$ and $\chi_{=}(K_n) = \chi^*_{=}(K_n) = n$. So $n = \chi_{=}(K_n) \leq \chi_{=}(K_n \cup P_n) \leq \max\{n, 2\}$. Hence $\chi_{=}(K_n \cup P_n) = n$.

Corollary 6.$\chi_{=}(K_n \cup C_m) = n$.

Proof. It is easy to verify that $\chi_{=}(K_n) = \chi^*_{=}(K_n) = n$, $\chi_{=}(C_m) = \chi^*_{=}(C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$. By the above theorem, $\chi_{=}(K_n \cup C_m) = n$.

Corollary 7.$\chi_{=}(K_n \cup K_n) = n$.

Proof. It is easy to verify that $\chi_{=}(K_n) = \chi^*_{=}(K_n) = n$. So $n = \chi_{=}(K_n) \leq \chi_{=}(K_n \cup K_n) \leq \max\{n, n\}$. Hence $\chi_{=}(K_n \cup Kn=n$.

IV. CONCLUSION

We have attempted to derive a generalized formula for the equitable chromatic number of the union of any two graphs in this study. The equitable chromatic number of the complete graph union path, complete graph union cycle, cycle union cycle, path union path, and complete graph union complete graph were also found.

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