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Functional Differential Inclusion of Fractional Order in Banach Algebras

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Abstract: In this paper, the existence of the solution for fractional order neutral functional differential inclusion in Banach algebras is proved. existence the extremal solution for fractional order neutral functional differential inclusion in Banach algebras is established under certain monotonicity conditions.

Keywords: *Functional Differential Inclusion, extremal solution, fixed point theorem*

1.1 Introduction:

In recent years, the subject of fractional calculus has attracted the great attention of mathematicians and a variety of results are available in literature. It is mainly due to its numerous applications in the fields of Physics, Mechanics, Chemistry, and Engineering [1, 13, 14]. Initial and boundary value problems of fractional order integral and differential equations have been extensively studied by several researchers. However, the same for integral and differential inclusions are rare. Integral and differential inclusions are regarded as generalizations of differential equations and inequalities (see Aubin Cellina et. al. [12], K. Deimling [15]). Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, and stochastic analysis, and so are widely studied by many authors ([16, 18, 19, 22]). Neutral functional differential equations is an important topic of functional differential equations and an exhaustive treatment may be found in Benchohra et. al. [17, 18].

The fixed-point theory for multivalued mappings is an important topic for multivalued analysis. Several well-known fixed point theorems of single-valued mappings such as those of Banach and Schauder have been extended to multi-valued mappings. The hybrid fixed point theorems for multi-valued mappings are very useful in proving the existence results under mixed Lipschitz and Caratheodory conditions.

In this paper, we shall study the existence of solution for a neutral functional differential inclusion of fractional order under the mixed Lipschitz and Caratheodory conditions. The main tools used in the study are the fixed point theorems [6, 7].

Statement of the problem:

Let $\mathbb{R}$ denote the real line. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Consider the fractional order neutral functional differential inclusion (in short FNFDI)

$$\left.\begin{array}{r} D^{\xi}\big(x(t) - f(t, x_t)\big) \in G(t, x_t) \quad a.e.\, t \in I \\ x_0 = \varphi(t) \ \ t \in I_0 \end{array}\right\} \qquad (1.1.1)$$

Where $f: I \times C \longrightarrow \mathbb{R}^n$ and $G: I \times C \longrightarrow \mathcal{P}(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ denotes the class of all non-empty subsets of $\mathbb{R}^n$.($\mathbb{R}^n$ be an n-dimensional Euclidean space). For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

We define a norm $|\cdot|$ in $\mathbb{R}^n$ by

$$|x| = |x_1| + \ldots + |x_n|$$

And $C = C(I_0, \mathbb{R}^n)$ denote the Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$ given by

$$\|\varphi\|_C = sup\{|\varphi(\theta)|; -r \le \theta \le 0\}$$

For any continuous function x defined on the interval $\mathbb{J} = [-r, a] = I_0 \cup I$ and $t \in I$ we denote by x_t the element of C defined by

$$x_t(\theta) = x(t+\theta) \qquad -r \leq \theta \leq 0, 0 \leq t \leq a$$

Where the function $\varphi \in C$.

Let us start by collecting some preliminary and important basic definitions and auxiliary results that will be used in the sequel.

1.2 Preliminaries

Fractional Calculus (see [1, 11, 20])

Definition 1.2.1 [3]: Let X be a Banach space. An operator $T: X \longrightarrow X$ is called compact if for any bounded subset S of X, $T(S)$ is a relatively compact subset of X. If T is continuous and compact, then it is called completely continuous on X.

Definition 1.2.2[6]: Let X be a Banach space. A mapping $T: X \longrightarrow X$ is called Lipschitz if there exists a constant $\alpha > 0$ such that, $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then T is called contraction on X with the contraction constant α.

Definition 1.2.3[20]: The Riemann-Liouville fractional integral of order $\xi > 0$ of a continuous function $f \in L^1[0, T]$ is defined by

$$I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} f(s) ds$$

Provided the right-hand side is point-wise defined on $(0, \infty)$

Definition 1.2.4 [20]: The Riemann-Liouville fractional derivative of order $\xi > 0$ of the function $g \in L^1[0, T]$ defined by

$$D^\xi g(t) = \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\xi-1} g(s) ds$$

$$n - 1 < \xi < n$$

Where Γ denotes the Euler gamma function.

Theorem 1.2.1 [10]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 1.2.2[10]: If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Multivalued Analysis (see [15, 21])

Let $\mathcal{P}(X)$ denote the class of all nonempty subsets of X. And $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cp}(X)$ and $\mathcal{P}_{cv}(X)$ denote respectively, the classes of all closed, bounded, compact, and convex subsets of X.

Definition 1.2.5 [9]: A mapping $Q: X \longrightarrow \mathcal{P}(X)$ is called a multivalued mapping or a multivalued operator on X and a point $u \in X$ is called a fixed point of Q if $u \in Qu$.

Let $(X, \|\cdot\|)$ be a norm space, $\mathcal{P}(X)$ denote the set of all non-empty subsets of X.

Let $Q: X \longrightarrow \mathcal{P}(X)$ be a multivalued map then

Definition 1.2.6 [21]: Q is convex valued if $Q(X)$ is convex for all $x \in X$.

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Definition 1.2.7 [15]: Q is bounded on a bounded set if $Q(Y) = \bigcup_{x\in Y} Q(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$, that is $sup_{x\in Y}\{sup\{|y|: y \in Q(x)\}\} < \infty$.

Definition 1.2.8 [15]: Q is upper semi-continuous (u. s. c.) on X if for each $x_0 \in X$, the set $Q(x_0)$ is a closed subset of X, and if for each open set N of X containing $Q(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $Q(\mathcal{N}_0) \subset \mathcal{N}$.

Definition 1.2.9 [21]: Q is compact if $\overline{Q(X)}$ is a compact subset of X.

Definition 1.2.10 [15, 21]: Q is totally bounded if for any bounded subset S of X, $Q(S) = \bigcup_{x\in S} Qx$ is totally bounded subset of X.

It is clear that every compact multivalued operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of X.

Definition 1.2.11 [15, 21]: Q is completely continuous if it is upper semicontinuous and compact multivalued operator on X.

Definition 1.2.12 [15, 21]: Q is completely continuous if $Q(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$;

Definition 1.2.13 [15, 21]: If the multi-valued map Q is completely continuous with nonempty compact values, then, Q is u.s.c. if and only if Q has a closed graph, that is, $x_n \to x_*, y_n \to y_*, y_n \in Q(x_n)$ imply that $y_* \in Q(x_*)$.

Definition 1.2.14 [15, 21]: The multi-valued map Q has fixed point if there is $x \in X$ such that $x \in Q(x)$. The fixed point set of Q will be denoted by $FixQ$.

The following Lemma is used in sequel

Lemma 1.2.1 [15]: If $G: X \longrightarrow \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$ that is, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \longrightarrow \infty$, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph then it is upper semi-continuous.

Lemma 1.2.2 [2]: Let X be a Banach space. Let $F: \mathbb{J} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(X)$ be an L^1- Caratheodory multivalued map and let Θ be a linear continuous mapping from $L^1(\mathbb{J}, X)$ to $C(\mathbb{J}, X)$. Then, the operator $\Theta \circ S_F: C(\mathbb{J}, X) \longrightarrow \mathcal{P}_{cp,cv}(C(\mathbb{J}, X))$, $x \longmapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(\mathbb{J}, X) \times C(\mathbb{J}, X)$.

The following theorem due to Dhage is used in the proof of existence the solution.

Theorem 1.2.3 [8]: Let X be a Banach algebra and let $\mathcal{A}: X \to \mathcal{P}_{cl,cv,bd}(X)$ and $\mathcal{B}: X \to \mathcal{P}_{cp,cv}(X)$ be multi-valued operators, satisfying

(a) $\mathcal{A}$ is contraction with contraction constants k

(b) $\mathcal{B}$ is completely continuous

Then either

I. The operator inclusion $\lambda x \in \mathcal{A}x + \mathcal{B}x$, has a solution for $\lambda = 1$ or

II. The set $\varepsilon = \{u \in X, \lambda u \in \mathcal{A}u + \mathcal{B}u, \lambda > 1\}$ is unbounded.

1.3 Existence Theory

Let $\mathbb{R}$ denote the real line. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. $\mathbb{R}^n$ be an n-dimensional Euclidean space. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We define a norm $|\cdot|$ in $\mathbb{R}^n$ by

$$|x| = |x_1| + \ldots + |x_n|$$

And $C = C(I_0, \mathbb{R}^n)$ denote the Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$ given by

$$\|\varphi\|_C = sup\{|\varphi(\theta)|: -r \le \theta \le 0\}$$

For any continuous function x defined on the interval $\mathbb{J} = [-r, a] = I_0 \cup I$ and $t \in I$ we denote by x_t the element of C defined by

$$x_t(\theta) = x(t + \theta) \qquad -r \le \theta \le 0, 0 \le t \le a$$

Where the function $\varphi \in C$.

We seek the solution of (1.1.1) in the space $C(\mathbb{J}, \mathbb{R}^n)$ of continuous real valued functions defined on $\mathbb{J}$. Define a norm $\|\cdot\|$ and a multiplication "." in $C(\mathbb{J}, \mathbb{R}^n)$ by

$$\|x\| = sup_{t \in \mathbb{J}} |x(t)|$$

and $\quad (x.y)(t) = (xy)(t) = x(t)y(t)$, $t \in \mathbb{J}$ for all $x, y \in C(\mathbb{J}, \mathbb{R}^n)$.

Then $C(\mathbb{J}, \mathbb{R}^n)$ is Banach algebra w. r. t. the above norm and multiplication in it.

By $L^1(\mathbb{J}, \mathbb{R}^n)$ we denote the space of Lebesgue-integrable function on $\mathbb{J}$ with the norm $\|\cdot\|_{L^1}$ defined by $\|x\|_{L^1} = \int_0^1 |x(t)| dt$

Definition 1.3.1 [5]: A multivalued mapping $G: \mathbb{J} \times C \to \mathcal{P}(\mathbb{R}^n)$ is said to be Caratheodory if

(i) $\quad t \mapsto G(t, x)$ is measurable for each $x \in C$;

(ii) $\quad x \mapsto G(t, x)$ is upper semicontinuous for almost all $t \in \mathbb{J}$;

Further a Caratheodory function G is called L^1-Caratheodory if

(iii) For each real number $r > 0$ there exists a function $h_r \in L^1(\mathbb{J}, \mathbb{R}^n)$ such that $\|G(t, x)\| = sup\{|v|: v \in G(t, x)\} \le h_r(t)$ a. e. $t \in \mathbb{J}$ for all $x \in C$ with $\|x\| \le r$

Finally, a Caratheodory function G is called L_X^1 -Caratheodory if

(iv) There exists a function $h \in L^1(\mathbb{J}, \mathbb{R}^n)$ such that $\|G(t, x)\| = sup\{|v|: v \in G(t, x)\} \le h(t)$ a. e. $t \in \mathbb{J}$ for all $x \in C$.

Definition 1.3.2 [45, 21]: For each $x \in C$, define the set of selections of G by

$$S_{G,x} := \{v \in L^1(\mathbb{J}, \mathbb{R}^n): v(t) \in G\big(t, x(t)\big) \, for \, a.e. \, t \in \mathbb{J}\}.$$

Definition 1.3.3: A function $x \in C(\mathbb{J}, \mathbb{R}^n)$ is said to be a solution of the neutral FNFDI (1.1.1) if there exists a $g \in L^1(\mathbb{J}, \mathbb{R}^n)$ with $g(t, x_t) \in G(t, x_t)$ a. e. on $\mathbb{J}$ such that

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$$D^{\xi}\big(x(t) - f(t, x_t)\big) = g(t, x_t) \tag{1.3.1}$$

and

(i) $x(t) = \varphi(t)$ if $t \in I_0$

(ii) $x_t \in C$ for $t \in I$

Lemma 1.3.1: The solution $x \in C(\mathbb{J}, \mathbb{R}^n)$ of neutral FNFDI (1.1.1) satisfies equation (1.3.1) if and only if it is a solution of the integral equation

$$\left.\begin{array}{c} x(t) = [\varphi(0) - f(0, \varphi)] + f(t, x_t) + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t \dfrac{G(x, x_s)}{(t-s)^{1-\xi}} ds \\ x(t) = \varphi(t) \;\; if \; t \in I_0 \end{array}\right\} \tag{1.3.2}$$

Proof: Apply I^{ξ} on (1.3.1)

$$I^{\xi} D^{\xi}\big(x(t) - f(t, x_t)\big) = I^{\xi} g(t, x_t)$$

$$[x(t) - f(t, x_t)]_0^t = \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(s, x_s)}{(t-s)^{1-\xi}} ds$$

$$[x(t) - f(t, x_t)] - [x(0) - f(0, x_0)] = \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(s, x_s)}{(t-s)^{1-\xi}} ds$$

$$x(t) = [\varphi(0) - f(0, \varphi)] + f(t, x_t) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(x, x_s)}{(t-s)^{1-\xi}} ds \tag{1.3.3}$$

Conversely differentiate (1.3.3) w. r. t. t to order ξ

$$D^{\xi} x(t) = D^{\xi}[\varphi(0) - f(0, \varphi)] + D^{\xi} f(t, x_t) + D^{\xi} \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(x, x_s)}{(t-s)^{1-\xi}} ds$$

We get $D^{\xi}\big(x(t) - f(t, x_t)\big) = g(t, x_t)$

Whence neutral FNFDI (1.1.1) is equivalent to integral inclusion

$$\left.\begin{array}{c} x(t) \in [\varphi(0) - f(0, \varphi)] + f(t, x_t) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{G(x, x_s)}{(t-s)^{1-\xi}} ds \\ x(t) = \varphi(t) \;\; if \; t \in I_0 \end{array}\right\} \tag{1.3.4}$$

1.4 Main Result

Existence the solution for FNFDI (1.1.1)

Let $X = C(\mathbb{J}, \mathbb{R}^n)$ be a Banach algebra

Define two operators $\mathcal{A}: X \longrightarrow \mathcal{P}_{cl,cv,bd}(X)$ by

$$\mathcal{A}x(t) = \begin{cases} \{-f(0, \varphi) + f(t, x_t)\} \; if \; t \in I \\ 0, if \; t \in I_0 \end{cases} \tag{1.4.1}$$

The multivalued operator $\mathcal{B}: X \longrightarrow \mathcal{P}_{cp,cv}(X)$ by

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$$\mathcal{B}x(t) = \begin{cases} u \in X; u(t) = \varphi(0) + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t \frac{g(s)}{(t-s)^{1-\xi}} ds \quad g \in S_{G,x} \quad if\ t \in I \\ \varphi(t)\ if\ t \in I_0 \end{cases} \quad (1.4.2)$$

Where, $S_{G,x} := \{g \in L^1(I, \mathbb{R}^n) : g(t) \in G(t, x_t)\ for\ a.e.\ t \in I\}$

Then the neutral FNFDI (1.1.1) is equivalent to the operator inclusion,

$$x(t) \in \mathcal{A}x(t) + \mathcal{B}x(t) \qquad t \in \mathbb{J} \qquad (1.4.3)$$

We shall discuss the operator inclusion (1.4.3) for the existence theorems under some suitable conditions on the functions and the multi-functions involved in it.

Consider the following hypothesis in the sequel.

(H$_1$) The function $f: \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a bounded function $k(t)$ with bound $\|K\|$ such that $|f(t,x) - f(t,y)| \leq k(t)\|x - y\|_C$ a. e. $t \in I$ and $\forall x, y \in C$. And $\|K\| < 1$.

(H$_2$) The multivalued function $G(t,x)$ has compact and convex values for each

$(t,x) \in I \times C$

(H$_3$) The multivalued function G is L^1-Caratheodory.

(H$_4$) There exists a bounded function h such that $h \in L^1(\mathbb{J}, \mathbb{R}^n)$ such that $\|G(t,x)\| \leq h(t)$ for all $t \in \mathbb{J}$.

Theorem 1.4.1: Assume that ((H$_1$) -(H$_4$) hold suppose that there exists a positive real number R such that,

$$R \leq \frac{F + \dfrac{\|h\|_{L^1} a^{\xi}}{\Gamma(\xi+1)}}{1 - \|K\|} \qquad (1.4.4)$$

Where $F = \|\varphi\|_C + |\varphi(0) - f(0,\varphi)| + sup_{t \in \mathbb{J}} |f(t,0)|$ then the FNFDI (1.1.1) has a solution on $\mathbb{J}$.

Proof: Let $X = C(\mathbb{J}, \mathbb{R}^n)$,we shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of theorem (1.2.3). For convenience we will split the proof in several steps

Step I: $\mathcal{A}$ has closed, convex and bounded values on X since by definition $\mathcal{A}x$ has singleton values for each $x \in X$. Now we will show that $\mathcal{A}$ has bounded values for bounded sets in X. To show this let S be a bounded subset of X then there exists $\sigma > 0$ such that $\|x\| \leq \sigma\ \forall x \in S$.

Then, for any $x \in S$, we have

$$\|\mathcal{A}x\| = \|\mathcal{A}x - \mathcal{A}0 + \mathcal{A}0\|$$

$$\|\mathcal{A}x\| \leq \|\mathcal{A}x - \mathcal{A}0\| + \|\mathcal{A}0\|$$

$$\|\mathcal{A}x\| \leq \|K\|\|x\| + \|\mathcal{A}0\|$$

$$\|\mathcal{A}x\| \leq \|K\|\sigma + \|\mathcal{A}0\|$$

Hence $\mathcal{A}$ is bounded on bounded subsets of X

Step II: Next, we will show that $\mathcal{A}$ is a contraction on X

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By (H_1)

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x_t) - f(t, y_t)|$$

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq k(t)\|x_t - y_t\|_C$$

Taking supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \|K\|\|x - y\|$$

This shows that $\mathcal{A}$ is a multivalued contraction since $\|K\| < 1$.

Next, we have to show that the multivalued operator $\mathcal{B}$ is completely continuous on X. A multivalued map is completely continuous if it is compact and upper semicontinuous (see Dhage [7]. To prove $\mathcal{B}$ to be compact using Arzela Ascoli theorem, we must show that $\mathcal{B}$ is uniformly bounded and equicontinuous. A multivalued map is upper semicontinuous if we establish that it has a closed graph. That can be proved in following steps.

Step III: To show that $\mathcal{B}$ maps bounded sets into bounded sets in X. To see this let S be a bounded set in X as defined above.

Now for each $u \in \mathcal{B}x$, there exists a $g \in S_{G,x}$ such that

$$u(t) = \varphi(0) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds$$

Then for each $t \in I$

$$|u(t)| = \left|\varphi(0) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds\right|$$

$$|u(t)| \leq |\varphi(0)| + \left|\frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds\right|$$

This further implies that

$$\|u\| \leq \|\varphi\|_C + \frac{1}{\Gamma(\xi+1)}\|h\|_{L^1}(a)^{\xi}$$

For all $u \in \mathcal{B}x \subset \cup\mathcal{B}(S)$. Hence $\cup\mathcal{B}(S)$ is bounded.

Next, we will show that $\mathcal{B}$ maps bounded sets into equicontinuous sets. Let S be as above, a bounded set and $u \in \mathcal{B}x$ for some $x \in S$ then there exists a $g \in S_{G,x}$ such that

$$u(t) = \varphi(0) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds$$

Then for any $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$ we have

$$|u(\tau_1) - u(\tau_2)| \leq \frac{1}{\Gamma(\xi)}\left|\int_0^{\tau_1} \frac{g(s)}{(\tau_1-s)^{1-\xi}}ds - \int_0^{\tau_2} \frac{g(s)}{(t_2-s)^{1-\xi}}ds\right|$$

$$|u(\tau_1) - u(\tau_2)| \leq \frac{\|h\|_{L^1}}{\Gamma(\xi)}\left|\int_0^{\tau_1} (\tau_1-s)^{\xi-1}ds - \int_0^{\tau_2} (\tau_2-s)^{\xi-1}ds\right|$$

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$$|u(\tau_1) - u(\tau_2)| \le \frac{\|h\|_{L^1}}{\xi\Gamma(\xi)}\left|\left[(\tau_1 - s)^\xi\right]_0^{\tau_1} - \left[(\tau_2 - s)^\xi\right]_0^{\tau_2}\right|$$

$$|u(\tau_1) - u(\tau_2)| \le \frac{\|h\|_{L^1}}{\Gamma(\xi+1)}\left|\left[(\tau_1 - s)^\xi\right]_0^{\tau_1} - \left[(\tau_2 - s)^\xi\right]_0^{\tau_2}\right|$$

$$|u(\tau_1) - u(\tau_2)| \le \frac{\|h\|_{L^1}}{\Gamma(\xi+1)}|{\tau_2}^\xi - {\tau_1}^\xi|$$

If $\tau_1, \tau_2 \in I_0$ then $|u(\tau_1) - u(\tau_2)| = |\varphi(\tau_1) - \varphi(\tau_2)|$

For the case when $\tau_1 \le 0 \le \tau_2$

$$|u(\tau_1) - u(\tau_2)| = \left|\varphi(\tau_1) - \left(\varphi(0) + \frac{1}{\Gamma(\xi)}\int_0^{\tau_2}\frac{g(s)}{(\tau_2 - s)^{1-\xi}}ds\right)\right|$$

$$|u(\tau_1) - u(\tau_2)| = |\varphi(\tau_1) - \varphi(0)| + \left|\frac{1}{\Gamma(\xi)}\int_0^{\tau_2}\frac{g(s)}{(\tau_2 - s)^{1-\xi}}ds\right|$$

$$|u(\tau_1) - u(\tau_2)| = |\varphi(\tau_1) - \varphi(0)| + \frac{\|h\|_{L^1}}{\Gamma(\xi)}\left|\int_0^{\tau_2}(\tau_2 - s)^{\xi-1}ds\right|$$

$$|u(\tau_1) - u(\tau_2)| = |\varphi(\tau_1) - \varphi(0)| + \frac{\|h\|_{L^1}}{\Gamma(\xi+1)}|{\tau_2}^\xi|$$

Hence, in all cases we have

$|u(\tau_1) - u(\tau_2)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$

As a result, $\cup\mathcal{B}(S)$ is an equicontinuous set in X. Now by Arzela-Ascoli theorem $\mathcal{B}$ is compact on X.

Step IV: Next, we prove that $\mathcal{B}$ has a closed graph.

Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in \mathcal{B}(x_n)$ and $y_n \longrightarrow y_*$ we shall prove that $y_* \in \mathcal{B}(x_*)$. Now since $y_n \in \mathcal{B}(x_n)$ $\exists$ a $g_n \in S_{G,x_n}$

Such that

$$y_n(t) = \begin{cases}\varphi(0) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g_n(s)}{(t-s)^{1-\xi}}ds \text{ if } t \in I \\ \varphi(t) \text{ if } t \in I_0\end{cases}$$

Consider a continuous linear operator $\Theta: L^1(X) \longrightarrow C(X)$ defined by

$$\Theta v(t) = \frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds$$

Now,

$$\|y_n(t) - y_*(t)\| = \left\|y_n(t) - \varphi(0) - \left(y_*(t) - \varphi(0)\right)\right\|$$

$$= \left\| \frac{1}{\Gamma(\xi)} \int_0^t \frac{g_n(s)-g_*(s)}{(t-s)^{1-\xi}} ds \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

From Lemma (1.2.2) it follows that $\Theta \circ S_{G,x}$ is a closed graph operator. Also, from the definition of Θ we have

$y_n - \varphi(0) \in \Theta \circ S_{G,x_*}$

Since $x_n \longrightarrow x_*$ and $y_n \longrightarrow y_*$ there is a $g_* \in S_{G,x_*}$ such that

$$y_*(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t \frac{g_*(s)}{(t-s)^{1-\xi}} ds \ if \ t \in I \\ \varphi(t) \ if \ t \in I_0 \end{cases}$$

As a result, $\mathcal{B}$ is an upper semi-continuous operator on X.

Thus $\mathcal{B}$ is an upper semi-continuous and compact and hence is completely continuous multi-valued operator on X.

Step V: Next, we will prove that $\mathcal{B}x$ is a convex subset of X for each $\in X$.

Let $u_1, u_2 \in \mathcal{B}x$. then there exists $g_1, g_2 \in S_{G,x}$ such that

$$u_i(t) = \varphi(0) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g_i(s)}{(t-s)^{1-\xi}} ds$$

For $i = 1,2$

Now for any $\theta \in [0,1]$ we have

$$\theta u_1(t) + (1-\theta)u_2(t) = \theta\varphi(0) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{\theta g_1(s)}{(t-s)^{1-\xi}} ds + (1-\theta)\varphi(0) \frac{1}{\Gamma(\xi)} \int_0^t \frac{(1-\theta) g_2(s)}{(t-s)^{1-\xi}} ds$$

$$\theta u_1(t) + (1-\theta)u_2(t) = \varphi(0) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{\theta g_1(s) + (1-\theta) g_2(s)}{(t-s)^{1-\xi}} ds$$

Since $G(t, x_t)$ is convex $\theta g_1(t, x_t) + (1-\theta) g_2(t, x_t) \in G(t, x_t)$ for all $t \in \mathbb{J}$

And so $\theta u_1 + (1-\theta)u_2 \in \mathcal{B}x$ which proves the convexity of $\mathcal{B}$.

As a result, $\mathcal{B}$ defines multivalued operator $\mathcal{B}: X \longrightarrow \mathcal{P}_{cp,cv}(X)$.

Step VI: Finally, we show that the set $\varepsilon = \{u \in X, \lambda u \in \mathcal{A}u + \mathcal{B}u, \lambda > 1\}$ is bounded

Let $u \in \varepsilon$ be any element then there exists a $g \in S_{G,x}$ such that

$$\lambda u(t) = [\varphi(0) - f(0,\varphi)] + f(t, u_t) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(s)}{(t-s)^{1-\xi}} ds$$

$$u(t) = \lambda^{-1}[\varphi(0) - f(0,\varphi)] + \lambda^{-1} f(t, u_t) + \frac{\lambda^{-1}}{\Gamma(\xi)} \int_0^t \frac{g(s)}{(t-s)^{1-\xi}} ds$$

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$$|u(t)| = \left|\lambda^{-1}[\varphi(0) - f(0,\varphi)] + \lambda^{-1}f(t,u_t) + \frac{\lambda^{-1}}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds\right|$$

$$|u(t)| \leq \|\varphi\|_C + |\varphi(0) - f(0,\varphi)| + |f(t,u_t)| + \left|\frac{1}{\Gamma(\xi)}\int_0^t \frac{g(s)}{(t-s)^{1-\xi}}ds\right|$$

$$|u(t)| \leq \|\varphi\|_C + |\varphi(0) - f(0,\varphi)| + |f(t,u_t) - f(t,0)| + |f(t,0)| + \left|\frac{\|h\|_{L^1}}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}ds\right|$$

$$\|u(t)\| \leq F + \|K\|\|u\| + \frac{\|h\|_{L^1}a^{\xi}}{\Gamma(\xi+1)}$$

Where $F = \|\varphi\|_C + |\varphi(0) - f(0,\varphi)| + sup_{t\in\mathbb{J}}|f(t,0)|$

$$\|u\| \leq \frac{F + \frac{\|h\|_{L^1}a^{\xi}}{\Gamma(\xi+1)}}{1 - \|K\|}$$

Put $\|u\| = R$

$$R \leq \frac{F + \frac{\|h\|_{L^1}a^{\xi}}{\Gamma(\xi+1)}}{1 - \|K\|}$$

This implies that conclusion II of theorem (1.2.3) does not hold by (1.4.4) hence conclusion I holds and operator inclusion $x(t) \in \mathcal{A}x(t) + \mathcal{B}x(t)$ consequently, FNFDI (1.1.1) has a solution on $\mathbb{J}$. this completes the proof.

1.5 Existence the extremal solutions

In this section, we shall prove the existence of maximal and minimal solutions of the FNFDI (1.1.1) under suitable monotonicity conditions on the multi-functions involved in it.

Definition 1.5.1[30]: Let X be a Banach space, a closed and nonempty set $\mathcal{K}$ in X is called a cone if

i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$

ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$

iii) $\{-\mathcal{K}\} \cap \mathcal{K} = 0$, where 0 is the zero element of X.

and is called a positive cone if

iv) $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

and the notation $\circ$ denotes a multiplication composition in X.

We define order relation $" \leq "$ in $\mathbb{R}^n$ as follows.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be any two elements. Then by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. We equip the space $C(\mathbb{J}, \mathbb{R}^n)$ with the order relation $\leq$ definition by the cone $\mathcal{K}$ in $C(\mathbb{J}, \mathbb{R}^n)$ that is,

$$\mathcal{K} = \{x \in C(\mathbb{J}, \mathbb{R}^n), x(t) \geq 0, \forall t \in \mathbb{J}\}$$

The cone $\mathcal{K}$ is normal in $C(\mathbb{J}, \mathbb{R}^n)$(see [30])

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Let $a, b \in C(\mathbb{J}, \mathbb{R}^n)$ be such that $a \le b$. then by an order interval $[a, b]$ we mean a set of points in $C(\mathbb{J}, \mathbb{R}^n)$ given by

$$[a, b] = \{x \in C(\mathbb{J}, \mathbb{R}^n); a \le x \le b\} \qquad (1.5.1)$$

Let, $A, B \in \mathcal{P}_{cl}\big(C(\mathbb{J}, \mathbb{R}^n)\big)$ then by $A \le B$ we mean $a \le b$ for all $a \in A$ and $b \in B$. Thus $a \le B$ implies that $a \le b$ for all $b \in B$, in particular, if $A \le A$ then it follows that A is a singleton set.

For the existence of extremal solution of FNFDI (1.1.1), we need the following definitions.

Definition 1.5.2: A function $a \in C(\mathbb{J}, \mathbb{R}^n)$ is said to be a lower solution of FNFDI (1.1.1) if

$$\left.\begin{aligned} D^{\xi}\big(a(t) - f(t, a_t)\big) \le g(t) \quad a.e.\ t \in I \\ a_0 \le \varphi(t) \ \ t \in I_0 \end{aligned}\right\} \qquad (1.5.2)$$

For all $g \in L^1(I, \mathbb{R}^n)$ such that $g(t) \in G(t, a_t)$ almost everywhere $t \in I$.

Definition 1.5.3: A function $a \in C(\mathbb{J}, \mathbb{R}^n)$ upper solution b of the FNFDI (1.1.1) is defined as, a function $b \in C(\mathbb{J}, \mathbb{R}^n)$ is called an upper solution of FNFDI (1.1.1) if

$$\left.\begin{aligned} D^{\xi}\big(b(t) - f(t, b_t)\big) \ge g(t) \quad a.e.\ t \in I \\ b \ge \varphi(t) \ \ t \in I_0 \end{aligned}\right\} \qquad (1.5.3)$$

For all $g \in L^1(I, \mathbb{R}^n)$ such that $g(t) \in G(t, b_t)$ almost everywhere $t \in I$.

Definition 1.5.4[9]: A solution x_M of the FNFDI (1.1.1) is said to be maximal if x is any other solution of FNFDI (1.1.1) on $\mathbb{J}$, then we have $x(t) \le x_M$ for all $t \in \mathbb{J}$.

Similarly, a minimal solution of the FNFDI (1.1.1) is defined as a solution x_m of the FNFDI (1.1.1) is said to be minimal if x is any other solution of FNFDI (1.1.1) on $\mathbb{J}$, then we have $x(t) \ge x_M$ for all $t \in \mathbb{J}$.

Definition 1.5.5 [7]: Let X be an ordered Banach space. A mapping $Q: X \longrightarrow \mathcal{P}_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$ then we have that $Qx \le Qy$.

We need the following theorem for the proof of existence of extremal solution.

Theorem 1.5.1[4]: Let $[a, b]$ be an order interval in a Banach space and let $\mathcal{A}, \mathcal{B}: [a, b] \longrightarrow \mathcal{P}_{cl}(X)$ be two multivalued operators satisfying

(a) $\mathcal{A}$ is multivalued contraction,

(b) $\mathcal{B}$ is completely continuous,

(c) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, and

(d) $\mathcal{A}x + \mathcal{B}x \subset [a, b]$, $\forall\, x \in [a, b]$.

Further if the cone $\mathcal{K}$ in X is normal, then the operator inclusion $x(t) \in \mathcal{A}x(t) + \mathcal{B}x(t)$ has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. Moreover $x_* = lim_n x_n$ and $x^* = lim_n y_n$, where $\{x_n\}$ and $\{y_n\}$ are sequences in $[a, b]$ defined by

$x_{n+1} \in \mathcal{A}x_n + \mathcal{B}x_n, x_0 = a$ and $y_{n+1} \in \mathcal{A}y_n + \mathcal{B}y_n, y_0 = b$.

We consider the following assumption in a sequel

(B1) The function $f(t,x)$ is nondecreasing in x almost everywhere for $t \in \mathbb{J}$.

(B2) The multivalued function $G(t,x)$ is nondecreasing in x a. e. for $t \in \mathbb{J}$.

(B3) The FNFDI (1.1.1) has a lower solution a and an upper solution b with $a \leq b$.

Theorem 1.5.2: Assume that the hypothesis (H_1) -(H_4) and (B_1) -(B_3) holds. Then the FNFDI (1.1.1) has a minimal and maximal solution.

Proof: Let $X = C(\mathbb{J}, \mathbb{R}^n)$ and consider the order interval $[a, b]$ in X. It is obvious from the hypothesis that the order interval $[a, b]$ is well-defined.

Define two operators $\mathcal{A}, \mathcal{B}: [a,b] \longrightarrow \mathcal{P}_{cl}(X)$ as in (1.4.1) and (1.4.2) respectively. As shown in the poof of theorem (1.4.1), similarly it can be shown that the $\mathcal{A}$ is contraction and $\mathcal{B}$ is completely continuous on $[a, b]$ respectively.

Next, we will show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[a, b]$.

For that, let $x, y \in [a, b]$ be such that $x \leq y$, $x \neq y$.Now by (B_1), we have

$\mathcal{A}x(t) = 0 = \mathcal{A}y(t)$ for all $t \in I_0$
and

$\mathcal{A}x(t) =- f(0,\varphi) + f(t, x_t)$

$\mathcal{A}x(t) \leq- f(0,\varphi) + f(t, y_t)$

$\mathcal{A}x(t) \leq \mathcal{A}y(t)$ for all $t \in I$

Hence $\mathcal{A}x(t) \leq \mathcal{A}y(t)$ for all $t \in \mathbb{J}$

Similarly, from (B_1), we have,

$\mathcal{B}x(t) = \varphi(t) = \mathcal{B}y(t)$ for all $t \in I_0$

$$\mathcal{B}x(t) = u \in X; u(t) = \varphi(0) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(x, x_s)}{(t-s)^{1-\xi}} ds \qquad g \in S_{G,x} \ \ if\ t \in I$$

$$\mathcal{B}x(t) \leq u \in X; u(t) = \varphi(0) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g(y, y_s)}{(t-s)^{1-\xi}} ds \qquad g \in S_{G,y} \ \ if\ t \in I$$

$\mathcal{B}x(t) \leq \mathcal{B}y(t)$ for all $t \in I$

Hence $\mathcal{B}x(t) \leq \mathcal{B}y(t)$ for all $t \in \mathbb{J}$.

Thus $\mathcal{A}$ and $\mathcal{B}$ are monotone increasing on $[a, b]$

Now by hypothesis (B_3), for any $x \in [a, b]$, we have.

$a \leq \mathcal{A}a + \mathcal{B}a \leq \mathcal{A}x + \mathcal{B}x \leq \mathcal{A}b + \mathcal{B}b \leq b$

This proves that

$\mathcal{A}x + \mathcal{B}x \subset [a, b]$

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Thus, the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of theorem (1.5.1) thus the operator inclusion and consequently FNFDI (1.1.1) has a maximal and minimal solution. This completes the proof.

1.6 Example

Consider the problem

$$\left.\begin{aligned} D^{1/2}\left(x(t) - \frac{1}{3}sin2t\left(\frac{x}{1+x}\right)\right) \in G(t, x_t) \ \ t \in I \\ x_0 = t \quad t \in I_0 \end{aligned}\right\}$$

Where $D^{1/2}$ is Riemann-Liouville derivative of order $1/2$ on the closed interval $\mathbb{J} = I \cup I_0 = [-1,0] \cup [0,1] = [-1,1]$ and multivalued map is given by

$$G(t, x_t) = \left(\sqrt{t/5}\, sin^2 x, \frac{t}{2} + \frac{|sinx|}{7}\right)$$

Here

$$f(t, x_t) = \frac{1}{3}sin2t\left(\frac{x}{1+x}\right)$$

$$h(t) = t + \frac{1}{7}$$

$$|f(t, x_t) - f(t, y_t)| = \left|\frac{1}{3}sin2t\left(\frac{x}{1+x}\right) - \frac{1}{3}sin2t\left(\frac{y}{1+y}\right)\right|$$

$$|f(t, x_t) - f(t, y_t)| \le \frac{1}{3}sin2t|x - y|$$

$$k(t) = \frac{1}{3}sin2t$$

$$\|K\| = \frac{1}{3} < 1$$

$$\|h\| = \frac{8}{7}$$

$$F = \|\varphi\|_C + |\varphi(0) - f(0, \varphi)| + sup_{t\in\mathbb{J}}|f(t, 0)|$$

$$\|\varphi\|_C = sup\{|\varphi(\theta)|; -1 \le \theta \le 0\} = 0$$

$$|\varphi(0) - f(0, \varphi)| = |0 - 0| = 0$$

$$sup_{t\in\mathbb{J}}|f(t, 0)| = 0$$

$$F = 0$$

$$R \le \frac{F + \frac{\|h\|_{L^1} a^{\xi}}{\Gamma(\xi + 1)}}{1 - \|K\|}$$

$$R \leq \frac{0 + \frac{8/7}{\Gamma(1/2 + 1)}}{1 - 1/3}$$

$$R \leq \frac{\frac{1.14285}{0.88622}}{\frac{2}{3}}$$

$$R \leq \frac{\frac{1.14285}{0.88622}}{\frac{2}{3}}$$

$$R \leq \frac{1.28957}{0.66666}$$

$$R \leq 1.93437$$

Since all the conditions of the theorem (1.4.1) are satisfied hence the given problem has a solution on $\mathbb{J}$.

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