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Analysis of Controllability and Stability in Nonlinear Discrete Dynamical Systems Through Functional Analytic Methods

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Abstract - Infinite-dimensional systems are often employed to model various phenomena, including propagation and transport processes, as well as population dynamics such as reproduction, development, and extinction. In economic systems, delays are inherent due to the time lapse between decisions and their impacts. Similarly, in communication networks, data transmission involves a non-zero time interval between initiation and delivery. Sometimes, these delays arise from simplifications in the model. These systems are characterized by their energetics, which can be described using difference equations that incorporate historical data of the system. Numerous numerical methods exist to represent such systems.

Keywords*: Stability, Nonlinear analysis.*

1. THE STABILITY ANALYSIS OF LINEAR DYNAMICAL SYSTEMS WITH TIME-DELAYS

1.1 Motivation and Historical Overview

The presence of time-delay components in a system can lead to oscillations, instability, and reduced performance. In some cases, even a small delay can destabilize the system, while in others, a large delay can have the same effect. As the delay in a linear time-delay system increases, the system may undergo multiple transitions between stability and instability. If the delay is governed by a nonlinear function, the delayed output of a chaotic system might be stabilized. This makes the stability analysis of time-delayed dynamical systems a crucial area of research for both theorists and practitioners.

1.2 A Dissipative Dynamical Systems Approach to the Stability Analysis of Time-Delay Systems

To achieve asymptotic stability, concepts of dissipative and exponential dissipative systems are employed. The relationship between a linear dynamical system and a time generator with an effectively infinite lag time is explored. Time delay operators exhibit both quadratic supply rates and a retention function with an integral component similar to the Lyapunov-Krasovskii integral term. A well-known necessary condition for linear dynamical systems is derived from this relationship to replicate the original time delay system. The dissipativity properties of time delay operators are used to develop an approach for constructing Lyapunov-Krasovskii functionals. Similar results are observed in discrete-time systems.

1.3 Main Concepts

Engineers in fields such as electronics and mechanics must be adept at using nonlinear analytical techniques to analyze and design nonlinear dynamical systems. Although these techniques have advanced significantly since the mid-1990s, nonlinear control remains challenging. This section provides fundamental findings for nonlinear system analysis, highlighting the differences from linear systems. It explains the most important nonlinear feedback control techniques, offering an overview of the primary methods available. The discussion also aims to contextualize the use of each technique.

REVIEW OF LITERATURE

Jerzy Klamka et al. [1] investigated discrete nonlinear finite-dimensional 1D and 2D control systems with constant coefficients to address issues related to local restricted controllability. Using mapping theorems from nonlinear functional analysis and linear approximation methods, they developed and established the necessary conditions for restricted controllability. These controllability requirements, initially applicable to unconstrained discrete systems with restricted controls, were extended to both 1D and 2D discrete systems under restricted controls.

Arash Hassibi et al. [2] discussed dynamical systems driven by asynchronously occurring events. Despite the time period

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\(T \) being infinite, the event rates were considered to be limited. The importance of these systems is growing in the control sector due to advancements in digital and communication systems, including asynchronous control systems, distributed control systems, and parallelized numerical methods. The research presented an advanced Lyapunov-based theory for controlling dynamical systems by solving bilinear matrix inequality (BMI) or linear matrix inequality (LMI) problems. The effectiveness of this method was demonstrated through various examples.

3 STEERING CONTROL OF SEMI-LINEAR DISCRETE DYNAMICAL SYSTEMS

3.1 Introduction

Krabs investigated a general discrete dynamical system of the form \(x(t+1) = f(x(t), u(t)) \). They also developed a linear control mechanism that directs the system from a given initial condition to a desired final state. In this chapter, we examine a semi-linear difference equation system of this type.

$$x(t+1) = A(t)x(i) + B(t)u(t) + f\big(t, x(t)\big), x(0) = x_0, t \in N_0 \qquad (3.1.1)$$

and its linear system:

$$x(t+1) = A(t)x(t) + B(t)u(t), x(0) = x_0, t \in N_0 \qquad (3.1.2)$$

Here, $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$are series of series $n \times n$ and $n \times m$ matrices, correspondingly, and $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0 0}$ are series of control vectors in R^m, and state vectors in R^n correspondingly, $f(.,.): N_0 \times R^n \to R^n$ with regard to the second input, a nonlinear function that satisfies the Lipschitz

It is shown that under specific conditions, we can steer any beginning state x_0 of system (3.1.1) to the preferred outcome desired x_1 in N∈N_0 time steps.

3.2 Steering Control for Semi-Linear System

Theorem 3.2.1

The nonlinear system described in (3.1.1) can be steered from its initial state \(x_0 \) to the desired final state within \(N \) steps.

Proof:

According to Theorem 2.7.2, our linear system (3.1.2) can be controlled if \(\det W_r (0, N) \neq 0 \). Thus, an alternative control approach can be employed in place of (3.1.5) to achieve the desired state transition.

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u(j) + \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j))$$

We get,

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)B(j)^* \Phi(N, j+1)^* W_r(0,N)^{-1} \left\{ x_1 - \Phi(t,0)x_0 - \sum_{j=0}^{t-1} \Phi(N, j+1) f(j, x(j)) \right\} + \sum_{j=0}^{t-1} \Phi(t, j+1) f(j, x(j)) \qquad (3.2.1)$$

t = 0. x(0) = x_0 and at t = N, x(N) = x_1. With respect to (3), x_0 is used as a starting point for the nonlinear system.

4 CONTROLLABILITY OF LINEAR VOLTERRA SYSTEMS

4.1 Controllability Using Controllability Gramian

We establish the controllability of a linear Volterra system using the controllability Gramian.

Theorem 4.1
Consider $\{A(t)\}t\in N0${A(t)}t∈N0 and $\{B(t)\}t\in N0${B(t)}t∈N0 as real-time $n\times n$n×n and $n\times m$n×m matrix sequences, respectively. Let LL be the operator defined as follows (4-1).

The following statements are equivalent:

1. A non-autonomous Volterra system can be controlled over the interval [0,][0,N].
2. The range of LL is RnRn.
3. The range of $LL*$LL∗ is RnRn.
4. The determinant of the controllability Gramian (0,N)≠0W(0,N)□=0 (4-1.8).

Proof. There is a solution to the system (4.1) by

$$x(t) = Q_t x_0 + \sum_{i=0}^{t-1} Q_i B(t-i-1)\, u(t-i-1)$$

5 STABILITY USING (SP) MATRIX

5.1 Importance of the (SP) Matrix

In this paper, we explore the exponential stability of null solutions to nonlinear non-autonomous discrete dynamical systems. The general form of such a system is given by:

x(t+1) = g(t, x(t)), \quad t \in \mathbb{N}_0 \tag{5.1}

where $g:N0\times\Omega\to\Omega$g:N0×Ω→Ω, with $\Omega\subseteq Rn$Ω⊆Rn. The (SP) matrix, introduced by Xue and Guo [63], provides a framework for defining a continuous nonlinear function that satisfies $(t,0)=0$g(t,0)=0 for all $t\in N0$t∈N0.

Consider the specific form:

$g(t,x(t))=Ax(t)+f(t,x(t))$g(t,x(t))=Ax(t)+f(t,x(t))

where $x(t)\in\Omega$x(t)∈Ω, and $A\in SA$∈S, with SS defined as:

$S=\{A=(aij)n\times n:aij\geq 0,\sum j=1naij\leq 1,\forall i=1,2,\ldots,n\}$S={A=(aij)n×n:aij≥0,∑j=1naij≤1,∀i=1,2,…,n}

This is the (SP) matrix, and the function $f:N0\times\Omega\to\Omega$f:N0×Ω→Ω satisfies the inequality:

$\|f(t,x(t))\|\leq a(t)\|x(t)\|,t\in N0$‖f(t,x(t))‖≤a(t)‖x(t)‖,t∈N0

where $\sum a(t)$∑a(t) is a convergent sequence of positive integers.

5.2 Exponential Stability of Null Solution of Semi-Linear System

To establish the exponential stability of the system's null solution, we use the well-known fact that the zero solution is exponentially stable if the Jacobian $D(0)$Dg(0) of system (5.1) has a spectral radius strictly less than 1. This can be verified by computing the eigenvalues of the Jacobian.

A practical approach to determine whether a matrix is an (SP) matrix involves verifying the conditions outlined in the definition of the (SP) matrix. This method does not require evaluating the eigenvalues of the Jacobian, making it efficient for numerical calculations.

Asymptotic Stability and the (SP) Matrix

As reported by Xue and Guo [63], the null solution of the linear system:

x(t+1) = Ax(t), \quad t \in \mathbb{N}_0 \tag{5.2}

is asymptotically stable if and only if AA is an (SP) matrix. This result is elaborated in Section 2.6.3 and Theorem 2.6.6. When considering the perturbed system:

x(t+1) = g(t, x(t)) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{N}_0 \tag{5.1}

under appropriate restrictions on the nonlinear function ff, it is possible to show that the perturbed system's null solution is exponentially stable. This significant conclusion emphasizes the importance of the (SP) matrix in ensuring the stability of nonlinear non-autonomous discrete dynamical systems.

To demonstrate a key theorem regarding the exponential stability of the zero solution in semi-linear systems, we proceed with the following statement:

Theorem on Exponential Stability of Null Solution

Consider the linear system: \[x(t+1) = A x(t), \quad t \in \mathbb{N}_0 \tag{1} \]

where \(A \) is an \(n \times n \) matrix belonging to \(S \), a class of matrices defined as \(S = \{ A = (a_{ij})_{n \times n} : a_{ij} \geq 0, \sum_{j=1}^n a_{ij} \leq 1 \ \forall i = 1, 2, \ldots, n \} \). \(A \) is termed as an (SP) matrix within this context.

Theorem 5.2.1

Statement:

If \(A \in S \) is an (SP) matrix, then the zero solution of the system (1) is not only asymptotically stable but also exponentially stable.

Proof Outline:

1. Asymptotic Stability:

By the properties of (SP) matrices (as defined in the literature), the eigenvalues of \(A \) satisfy \(|\lambda_i| \leq 1 \) for all eigenvalues \(\lambda_i \). This property ensures that the zero solution of the system (1) is asymptotically stable. That is, for any \(\epsilon > 0 \), there exists \(\delta > 0 \) such that if \(\| x(0) \| < \delta \), then \(\| x(t) \| < \epsilon \) for all \(t \geq 0 \).

2. Exponential Stability:

To prove exponential stability, we establish that there exist constants \(\beta > 0 \) and \(\eta \in (0, 1) \) such that \(\| \Phi(t, 0) \| \leq \beta \eta^t \), where \(\Phi(t, 0) \) denotes the transition matrix of the system (1).

This follows from the fact that the norm of \(\Phi(t, 0) \) is bounded by \(\beta \eta^t \) due to the spectral properties of \(A \) as an (SP) matrix. Specifically, \(\eta \) relates to the spectral radius of \(A \) and \(\beta \) accounts for the initial conditions and the properties of \(A \).

Therefore, the theorem concludes that the zero solution of the linear system (1) with \(A \) being an (SP) matrix is exponentially stable. This result underscores the significance of (SP) matrices in ensuring both asymptotic and exponential stability of linear systems.

6 OPTIMAL CONTROL OF DISCRETE VOLTERRA SYSTEM - A CLASSICAL APPROACH

In this chapter, we explore the optimal control problem for discrete-time linear Volterra systems using the classical approach of Lagrange multipliers for minimization

6.1 Introduction

Many researchers have addressed optimal control problems in discrete Volterra systems. Gaishun and Dymkov employed operator techniques to analyze response control in linear discrete Volterra systems, while Belbas and Schmidt studied optimal management of Volterra integral equations with impulse components. This chapter focuses on optimizing control of linear Volterra systems using Lagrangian multipliers, a standard minimization technique.

$$x(t+1) = \sum_{i=0}^{t} A\,(i)x(t-i) + Bu(t), t \in N_0$$

Where A(i) is a n x n nonsingular matrix with I = 0,1...t and B is a n x m matrix. For the fixed time procedure (0 t N), we use a quadratic performance index.

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}[x^*(t)Qx(t) + u^*(t)Ru(t)]$$

Both S and Q are Hermitian matrices that have an nxn number of positive or positive semi-definite (or existent symmetric matrices). mxm A positive and definite hermitian or real-symmetric matrix A matrix of the Hermitian type A controller described by R Equation minimises J when applied to the constraint equation (and what time starting conditions on the state vector are as follows:

$$x(0) = c.$$

6.2 Solution by the Classical Minimization Method Using Lagrange Multipliers

As a result, we reduce J definite by

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}[x^*(t)Qx(t) + u^*(t)Ru(t)]$$

Assuming the control equation is true

$$x(t+1) = \sum_{i=0}^{t} A_i x\,(t-i) + Bu(t), t \in N_0$$

Here's how to represent the starting condition on the state vector: where t = 0, 1, 2,..., N - 1, and

$$x(0) = c$$

λ(1),λ(2),…λ(N),λ(i)^' s∈R^n, for i = 1,2,..., N,

When we take into account a set of Lagrange multipliers we arrive at L, which is a new performance index.

$$\begin{aligned} L = {} & \frac{1}{2}x^*(N)Sx(N) \\ & + \frac{1}{2}\sum_{t=0}^{N-1}\left\{[x^*(t)Qx(t) + u^*(t)Ru(t)]\right. \\ & + \lambda^*(t+1)\left[\sum_{i=0}^{t} A_i x\,(t-i) + Bu(t)\right. \\ & \left. - x(t+1)\right] \\ & + \left[\sum_{i=0}^{t} A_i x\,(t-i) + Bu(t)\right. \\ & \left.\left. - x(t+1)\right]^* \lambda(t+1)\right\} \end{aligned}$$

Evidently L = L*.

Then, we set the results equal to 0 by dividing L by every part of x(t), u(t), or A(t). i.e. To minimize the functional L,

$$\frac{\partial L}{\partial x(t)} = 0, t = 1,2, \dots, N$$

$$\frac{\partial L}{\partial x(t)} = 0, t = 0,1,2, \dots, N-1$$

$$\frac{\partial L}{\partial \lambda(t)} = 0, t = 1,2, \dots, N$$

As an example, consider the subsequent (submit Ogata, page 670),

$$\frac{\partial L}{\partial x}x^*Ax = 2Ax, if\ A = A^*,$$

And

$$\frac{\partial L}{\partial x}x^*Ay = Ay$$

There are a number of partial derivatives that

$$\frac{\partial L}{\partial x(t)} = 0$$

In other words,

$$Qx(t) + \sum_{i=0}^{N-t-1} A^*\,(i)\lambda(t+i+1) - \lambda(t) = 0, t = 1,2, \dots, N-1$$

Now

$$\frac{\partial L}{\partial x(N)} = 0$$

Implies

$$Sx(N) - \lambda(N) = 0$$

Similarly

$$\frac{\partial L}{\partial x(t)} = 0$$

$$Ru(t) + B^* \lambda(t+1) = 0$$

$$\frac{\partial L}{\partial \lambda(t)} = 0$$

$$\sum_{i=0}^{t-1} A(i)\, x(t-i-1) + Bu(t-1) - x(t) = 0$$

System equation calculates the Lagrange multiplier. In the next step, we will simplify the equations that we have just. As a result of Equation,

$$\lambda(t) = Qx(t) + \sum_{i=0}^{N-t-1} A^*(i)\lambda(t+i+1), t = 1,2,\dots,N-1$$

λ(N)=Sx(N) is the final condition

Solving for u(t) and noting that f?-1 occurs in (6.2.4) gives us

$$u(t) = -R^{-1}B^* \lambda(t+1), t = 0,1,\dots,N-1$$

It is possible to rewrite equation as

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t+i) + Bu(t), t = 0,1,\dots,N-1$$

What we have here is a state equation. The effect of replacing

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t+i) - BR^{-1}B^*\lambda(t+1), t = 0,1,\dots,N-1$$

With the first situation x(0) = c.

We must solve concurrently in order to find the answer to the minimization issue. Note that the system specifies the first state x(0), whereas the Lagrange multiplier equation specifies the ending condition X(N).Thus,

$$u(t) = -R^{-1}B^*\lambda(t+1), t = 0,1,\dots,N-1$$

6.3 Summary

The design of controllers for discrete-time linear Volterra systems was investigated using the classical Lagrange multiplier approach.

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