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A Unified Approach to Fixed Points in Fuzzy Metric Spaces

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Abstract: In this paper, we delve into the realm of fuzzy metric spaces to establish new fixed point results by leveraging the concepts of compatibility and weak compatibility among six self-mappings. This research extends the foundational work of Singh and Chouhan [13], broadening the scope and applicability of their results on common fixed points within fuzzy metric spaces. Through rigorous analysis and the introduction of novel techniques, we demonstrate the conditions under which these self-maps exhibit fixed points, thereby contributing to the theoretical advancement of fuzzy metric space theory. Our findings not only validate and extend existing results but also open new avenues for further exploration and application in mathematical and computational contexts where fuzziness and metric considerations are pivotal.

INTRODUCTION

The concept of Fuzzy sets was given by Zadeh [17]. Subsequently, several researchers in Analysis and Topology used it. The paper is dealt with the Fuzzy metric space defined by Kramosil and Michalek [11] and modified by George and Veeramani [5]. Grebiec [6] has proved fixed point results for Fuzzy metric space. In this connection, Singh and Chouhan [13] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Vasuki [15] proved the fixed point theorems using the concept of R-weak commutativity of mappings for Fuzzy metric space.

Recently, Jungck and Rhoades [10] introduced the concept of weak compatible maps. The concept is most general among all the commutativity maps. For this, every pair of R-weakly commuting self maps is compatible and each pair of compatible self maps is weakly compatible but the converse is not true.

A fixed-point theorem for six self maps using the concept of weak compatibility and compatibility of pairs of self-maps in fuzzy metric space has been proved in this paper. The result of Singh and Chouhan [13] has been generalized.

For this, we need the following definitions and Lemmas.

2. PRELIMINARIES

Definition 2.1[2] A binary operation *: [0, 1] x [0, 1] $\rightarrow$ [0, 1] is called a t-norm if $([0,1]*)$ is an abelian topological monoid with unit 1 such that a * b ≤ c *d whenever a ≤ c and b ≤ d for a, b, c, d $\in$ [0, 1].

Examples of t-norms are a * b = ab and a * b = min $\{a, b\}$.

Definition 2.2 [9] The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions:

(FM-1) M (x, y, 0) = 0,

(FM-2) M (x; y, t) = 1 for all t > 0 if and only if x = y,

(FM-3) M (x, y, t) = M (y, x, t),

(FM-4) M (x, y, t) * M (y, z, s) ≤ M (x, z, t + s),

(FM-5) M (x, y,.): $[0, \infty) \rightarrow$ [0, 1] is left continuous,

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(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$.

for all x, y, z $\in$ X and s, t > 0.

Note that M (x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M (x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1 [2] Let (X, d) be a metric space. Define a * b = min {a, b} and M (x, y, t) = $\frac{t}{t + d(x, y)}$ for all x, y $\in$ X and all t > 0. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3 [3] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a Cauchy sequence if and only if for each ε > 0, t > 0, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all n, m $> n_0$.

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each ε > 0, t > 0 there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all n, m$\geq n_0$.

A Fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in it converges to a point in it.

Proposition 2.1 [13]Self mappings A and S of a Fuzzy metric space (X, M, *) are compatible then they are weakly compatible.

Proof. Suppose Ap = Sp, for some p in X. Consider a constant sequence $\{p_n\}$ = p. Now, $\{Ap_n\} \to Ap$ and $\{Sp_n\} \to Sp(= Ap)$.

As A and S are compatible we have $M(ASp_n, SAp_n, t) \to 1$ for all t > 0 as $n \to \infty$. Thus ASp = SAp and we get that (A, S) is weakly compatible.

The following is an example of pair of self maps in a Fuzzy metric space which are weakly compatible but not compatible.

Example 2.2 [9] Let (X, M, *) be a Fuzzy metric space where X= [0, 2]. t-norm is defined by a * b = min $\{a, b\}$ for all $a, b \in$ [0, 1] and $M(x, y, t) = e^{\frac{|x-y|}{t}}$ for all x, y $\in$ X. Define self maps A and S on X as follows:

$$Ax = \begin{cases} 2 - x & \text{if} \quad 0 \leq x < 1 \\ 2 & \text{if} \quad 1 \leq x \leq 2 \end{cases} \qquad \text{And}$$

$$Sx = \begin{cases} x & \text{if} \quad 0 \leq x < 1 \\ 2 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

Taking $x_n = 1 - \frac{1}{n}; \;\; n = 1, 2 3, \ldots$

Then $x_n \to 1, x_n < 1$ and $2 - x_n > 1$ for all n.

Also $Ax_n, Sx_n \to 1$ as $n \to \infty$. Now

$$M(ASx_n, SAx_n, t) = e^{\frac{|ASx_n - SAx_n|}{t}} \to e^{-\frac{1}{2}} \neq 1 \text{ as } n \to \infty.$$

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Hence the pair (A, S) is not compatible. Also set of coincident points of A and S is [1, 2].

Now for any $x \in [1, 2]$, Ax = 8x = 2 and AS(x) = A (2) = 2 = S (2) = SA(x). Thus A and S are weakly compatible but not compatible.

From the above example, it is obvious that the concept of weak compatibility is more general than that of compatibility.

Lemma 2.1 [1] Let $\{x_n\}$ be a sequence in a Fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$ for all and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2 [13] Let (X, M, *) be a Fuzzy metric space. If there exists $k \in (0,1)$ such that for all $x, y \in X$.

$M(x, y, kt) \geq M(x, y, t)$ for all $t > 0$, then x = y.

Theorem 2.1 [13] Let A, B, S, T, P and Q is self maps on a complete Fuzzy metric space (X, M, *) with * is a continuous t-norm for all t > 0 satisfying the following conditions

(a) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(b) AB = BA, ST = TS, PB = BP, QT = TQ;

(c) (P, AB) is compatible and (Q, ST) is weakly compatible;

(d) Either AB or P is continuous;

(e) There exists $k \in (0, 1)$ such that

$$M(Px, Qy, kt) \geq \min\{M(ABx, Px, t), M(STy, Qy, t), M(STy, Px, \beta t), M(ABx, Qy, (2-\beta)t) M(ABx, STy, t)\},$$

For all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$. Then A, B, S, T, P and Q have a unique common fixed point in X.

3. MAIN RESULT.

Theorem 3.1 Let A, B, S, T, P and Q be self-maps of a complete fuzzy metric space (X, M, *) with * is a continuous t-norm for all $t > 0$ satisfying the following conditions

(a) $P(x) \subseteq ST(x)$, $Q(x) \subseteq AB(x)$

(b) AB = BA, ST = TS, PB = BP, QT = TQ.

(c) (P, AB) is compatible, and (Q, ST) is weakly compatible

(d) Either AB or P is continuous.

(e) There exists $K \in (0,1)$ such that

$$M(Px, Qy, kt) \geq \min\left\{M(Qy, STy, t), M(ABx, STy, t), \frac{M(Px, ABx, t), M(Qy, STy, t)}{M(Px, Qy, t)}, M(Px, ABx, t)\right\}$$

For all $x, y \in x$ and $t > 0$. Then A, B, S, T, P and Q have a unique common fixed Point in X.

Proof. Let $x_0 \in X$ from (a) $\exists\, x_1, x_2 \in X$ such that

$$Px_0 = STx_1 = y_0 \quad \text{and} \quad Qx_1 = ABx_2 = y_1$$

Inductively, we can contract seq $\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n} = STx_{2n+1} = y_{2n} \text{ and } Qx_{2n+1} = ABx_{2n+2} = y_{2n+1} \qquad n = 0, 1, 2, ...$$

I. put $x = x_{2n}, y = x_{2n+1}$ for t > 0 in (e) we get

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \min\Big\{M(Qx_{2n+1}, STx_{2n+1}, t),\ M(ABx_{2n}, STx_{2n+1}, t), \frac{M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t)}{M(Px_{2n}, Qx_{2n+1}, t)}, M(Px_{2n}, ABx_{2n}, t)\Big\}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\Big\{M(y_{2n}, y_{2n+1}, t),\ M(y_{2n-1}, y_{2n}, t) \frac{M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t)}{M(y_{2n}, y_{2n+1}, t)}, M(y_{2n}, y_{2n-1}, t)\Big\} \quad \text{By (FM 3)}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\Big\{M(y_{2n+1}, y_{2n}, t),\ M(y_{2n-1}, y_{2n}, t) \frac{M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t)}{M(y_{2n+1}, y_{2n}, t)},\ M(y_{2n-1}, y_{2n}, t)\Big\}$$

$$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t),$$

$$M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t)\}$$

$$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t)\}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\{M(y_{2n-1}, y_{2n}, t),\ M(y_{2n}, y_{2n+1}, t)\}$$

Similarly

$$M(y_{2n+1}, y_{2n+2}, kt) \geq \min\{M(y_{2n}, y_{2n+1}, t),\ M(y_{2n+1}, y_{2n+2}, t)\}$$

Therefore for all n even or odd, we have

$$M(y_n, y_{n+1}, kt) \geq \min\{M(y_{n-1}, y_n, t),\ M(y_n, y_{n+1}, t)\}$$

$$M(y_n, y_{n+1}, t) \geq \min\{M(y_{n-1}, y_n, t/k),\ M(y_n, y_{n+1}, t/k)\}$$

Be repeating the above inequality, we have

$$M(y_n, y_{n+1}, t) \geq \min\{M(y_{n-1}, y_n, t/k), M(y_n, y_{n+1}, t/k^m)\}$$

Since $M(y_n, y_{n+1}, t/k^m) \to 1$ as $m \to \infty$, it follows that

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k)$$

i.e. $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $n \in N$ $t > 0$

By lemma 2.1, this implies that $\{y_n\}$ is Cauchy sequence in x. Since x is complete $\{y_n\} \to z \in X$. Also its subsequences converge to the same point i.e. $z \in X$.

$\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$

Firstly, suppose AB is continuous.

As AB is continuous, $(AB)^2 x_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$

As (P, AB) is continuous pair, we have $P(AB)x_{2n} \to ABz$

II. Putting $x = ABx_{2n}, y = x_{2n+1}$ in (e), we have

$$M(PABx_{2n}, Qx_{2n+1}, kt) \geq \min\Big\{M(Qx_{2n+1}, STx_{2n+1}, t),\ M(ABABx_{2n}, STx_{2n+1}, t),$$
$$\frac{M(PABx_{2n}, ABABx_{2n}, t), M(Qx_{2n+1}, ST_{2n+1}, t)}{M(PABx_{2n}, Qx_{2n+1}, t)},$$
$$M(PABx_{2n}, ABABx_{2n}, t)\Big\}$$

Letting $n \to \infty$, we get

$$\geq \min\Big\{M(z, z, t), M(ABz, z, t),$$
$$\frac{M(ABz, ABz, t), M(z, z, t)}{M(ABz, z, t)}, M(ABz, ABz, t)\Big\}$$

$$\geq \min\left\{M(ABz, z, t), \frac{1}{M(ABz, z, t)}\right\}$$

$$M(ABz, z, kt) \geq M(ABz, z, t)$$

Therefore by lemma 2.2 ABz = z

(III) Putting x = z, y = x_{2n+1} in (e) we get

$$M(Pz, Qx_{2n+1}, kt) \geq \min\Big\{M(Qx_{2n+1}, STx_{2n+1}, t),\ M(ABz, STx_{2n+1}, t)$$
$$\frac{M(Pz, ABz, t), M(Qx_{2n+1}, STx_{2n+1}, t)}{M(Pz, Qx_{2n+1}, t)}, M(Pz, ABz, t)\Big\}$$

Letting $n \to \infty$, we get

$$M(Pz, z, kt) \geq \min\Big\{M(z, z, t),\ M(ABz, z, t)\ \frac{M(Pz, ABz, t), M(z, z, t)}{M(Pz, z, t)},\ M(Pz, ABz, t)\Big\}$$

$$M(Pz,z,kt) \geq \min\left\{M(z,z,t),\ M(z,z,t)\ \frac{M(Pz,z,t),M(z,z,t)}{M(Pz,z,t)},\ M(Pz,z,t)\right\}$$

$$M(Pz,z,kt) \geq M(Pz,z,t)$$

Hence by lemma 2.2 Pz = z

therefore ABz = Pz = z

IV. Put x = Bz, y = x_{2n+1} in (e) we get

$$M(PBz,Qx_{2n+1},kt) \geq \min\left\{M(Qx_{2n+1},STx_{2n+1},t),\ M((AB)Bz,STx_{2n+1},t)\right.$$
$$\left.\frac{M(PBz,ABBz,t),M(Qx_{2n+1},STx_{2n+1},t)}{M(PBz,Qx_{2n+1},t)},\ M(PBz,ABBz,t)\right\}$$

As AB = BA, PB = BP, Letting $n \to \infty$, we get

$$M(PBz,z,kt) \geq \min\left\{M(z,z,t),\ M(B(AB)z,z,t)\right.$$
$$\left.\frac{M(BPz,B(AB)z,t),M(z,z,t)}{M(Bz,z,t)},\ M(Bz,Bz,t)\right\}$$

$$M(Bz,z,kt) \geq \min\left\{M(Bz,z,t),\frac{1}{M(Bz,z,t)}\right\} \quad M(Bz,z,kt) \geq M(Bz,z,t)$$

From by lemma 2.2 Bz = z As ABz = z $\Rightarrow$ Az = z

Therefore Az = Bz = Pz = z

V. As $P(X) \subseteq ST(X)$, there exist $v \in X$ such that z = Pz = STv

Putting $x = x_{2n}$, y = v in (e) we get

$$M(Px_{2n},Qv,kt) \geq \min\left\{M(Qv,STv,t),\ M(ABx_{2n},STv,t)\right.$$
$$\left.\frac{M(Px_{2n},ABx_{2n},t),M(Qv,STv,t)}{M(Px_{2n},Qv,t)},\ M(Px_{2n},ABx_{2n},t)\right\}$$

Letting $n \to \infty$ we get

$$M(z,Qv,kt) \geq \min\left\{M(Qv,z,t),\ M(z,z,t)\ \frac{M(z,z,t),M(Qv,z,t)}{M(z,Qv,t)},\ M(z,z,t)\right\} \text{ (by FM3)}$$

$$M(z,Qv,kt) \geq M(z,Qv,t)$$

By lemma 2.2 $Qv = z$. Hence STv = z = Qv

As (Q, ST) is weakly compatible we have STQv=QSTv

Thus STz = Qz

VI. Put $x = x_{2n}$, y = z in (e) we get

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$$M(Px_{2n}, Qz, kt) \geq \min\{M(Qz, STz, t),\ M(ABx_{2n}, STz, t)$$

$$\frac{M(Px_{2n}, ABx_{2n}, t), M(Qz, STz, t)}{M(Px_{2n}, Qz, t)},\ M(Px_{2n}, ABx_{2n}, t)\}$$

Letting $n \to \infty$ we get

$$M(z, Qz, kt) \geq \min\{M(Qz, Qz, t),\ M(z, Qz, t),\ \frac{M(z, z, t), M(Qz, Qz, t)}{M(z, Qz, t)},\ M(z, z, , t)\}$$

$$M(z, Qz, kt) \geq \min\left\{M(z, Qz, t), \frac{1}{M(z, Qz, t)}\right\}$$

$$M(z, Qz, kt) \geq M(z, Qz, t)$$

By Lemma 2.2 Qz = z = STz

VII. Put $x = x_{2n}$, y = Tz in (e) we get

$$M(Px_{2n}, QTz, kt) \geq \min\{M(Qz, STz, t),\ M(ABx_{2n}, STTz, t)$$

$$\frac{M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, t)}{M(Px_{2n}, QTz, t)}, M(Px_{2n}, ABx_{2n}, t)\}$$

Letting $n \to \infty$ and using condition (b) we get

$$M(Px_{2n}, QTz, kt) \geq \min\{M(z, z, t),\ M(z, Tz, t),\ \frac{M(z, z, t), M(Tz, Tz, t)}{M(z, Tz, t)},\ M(z, z, t)\}$$

$$M(z, Tz, kt) \geq \min\left\{M(z, Tz, t), \frac{1}{M(z, Tz, t)}\right\}$$

$$M(z, Tz, kt) \geq M(z, Tz, t)$$

From Lemma 2.2 Tz = z As STz = z $\Rightarrow$ Sz = z

Qz = Sz = Tz = z ... (B)

From (A) and (B) Az = Bz = Sz = Tz = Qz = Pz = z

Hence z is a common fixed point of A, B, S, T, P and Q.

Uniqueness.

Let u be another common fixed pt of A, B, S. T, P and Q Then

Au = Bu = Pu = Qu = Su = Tu = u

Put x = z, y = u in (e)

$$M(Pz, Qu, kt) \geq \min\{M(Qu, STu, t),\ M(ABz, STu, t)$$

$$\frac{M(Pz, ABz, t), M(Qu, STu, t)}{M(Pz, Qu, t)},\ M(Pz, ABz, t)\}$$

$$M(z,u,kt) \geq \min\left\{M(u,u,t),\ M(z,u,t)\ \frac{M(z,z,t), M(u,u,t)}{M(z,u,t)},\ M(z,z,t)\right\}$$

$$M(z,u,kt) \geq \min\left\{M(z,u,t), \frac{1}{M(z,u,t)}\right\}$$

$$M(z,u,kt) \geq M(z,u,t)$$

By Lemma 2.2 z = u

Hence z is a unique common fixed pt of self-maps A, B, S, T, P and Q.

BIBLIOGRAPHY

[1] Caristi, J.: Fixed point theorems for mapping satisfying inwardness conditions, Tran. Amer. Soc.
[2] 25(1976),241-251.
[3] Chaterjea, S.K.: Fixed paint theorem for a sequence of matting with a contraction iterates. Publ.
[4] Math. Inst. (Beograd) (N.S.), 14 (28) (1972), 15-18.
[5] Cho, Y.J.: Fixed Point in Fuzzy Metric Space J.Fuzzy Math. 5 (1997), 949-962.
[6] Edelstein, M.: On fixed point and periodic points under contractive mappings. Jour. London, Math. Soc. 37 (1906), 74-.George, A. and Veeramani, P.: On same results in Fuzzy metric spaces, Fuzzy Sets and System 64 (1994), 395-399.
[7] Grebiec, M.: Fixed points in Fuzzy metric spaces, Fuzzy sets and systems 27 (1998), 385-389.
[8] Hardy, G.E. and Rogers, T.O.: A generalization of a fixed point theorem of Riech, Canad. Math. Bull. 16 (1973), 201-206.
[9] Hu, T.K.: On a fixed point theorem for metric spaces. Amer. Math. Monthly 74(1967),446-437.
[10] Jungck, G.: Campatibte maps and common fixed points, Internat. Jour. Math. and Sci. a (1986), 771-779.
[11] Jungck,G. and Rnoades,B.E. Fixed points for set valued functions without containity, Indian J.Pure Appl.Math.29(1999),227-240.
[12] Kramosil I, and Michalek, J.: Fuzzy metic and statistaical metric spaces, Kybernetica 11 (1975), 336-344.
[13] Mishra, S.N., Mishra N and singh, S.L.: Common fixed point of maps in fuzzy metric space Int. J. Math. Math. Sci 17 (1994), 253
[14] Singh B.and Chouhan, M.S.: Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy sets and systems 115(2000), 471-475
[15] Vasuki, R.: Common Fixed point theorem in a fuzzy metric space, Fuzzy Sets and Systems 97 (1998), 395-397,
[16] Vasuki, R.: Common fixed point for R-Weakly commuting maps in Fuzzy metric spaces, Indian J. Pure Appl. Math 30 (4), (1999), 419-423.
[17] Weston, A.: Characterization of metric completeness. Proc. Amer. Math. Soc. 64 (1977), 186-188.
[18] Zadeh, L.A.: Fuzzy sets, Inform control 89 (1965), 338-353.