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Some Special Functions of Complex Variable

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Abstract: Main aim of this article is the discussion of Univalent complex functions, Cap like complex functions, and star like complex functions, close-to-cap like complex functions.

Keywords: *Univalent function; BEIRBARBACH Conjecture; Cap like function; Star like function; KOEBE's function; HADAMARD Product (Convolution). Close-to-Cap like functions.*

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Introduction: We know that a complex valued function is said to be regular or analytic in a domain **D** (a non-empty open connected subset of the complex plane $\mathbb{C}$) if it has a uniquely determined derivative at each point of **D.**

Definition 1: A function $\mathbf{f}(z)$ is said to be a univalent in a domain **D** if $\mathbf{f}(z_1)\neq\mathbf{f}(z_2)$ for all $\{z_1,z_2\}\subset\mathbf{D}$ with $z_1\neq z_2$.

A necessary condition for analytic function $\mathbf{f}(z)$ to be univalent in **D** is $\mathbf{f}'(z)\neq 0$ in **D.** This condition is not sufficient since $\mathbf{f}(z)=e^z$ is clearly not univalent since $\mathbf{f}(0)=e^0=1=e^{\imath 2\pi}=\mathbf{f}(\imath 2\pi)$ but $\mathbf{f}'(z)=e^z\neq 0$.

By Riemann mapping theorem, one function may map any simply connected domain onto the open unit disc in a one-one conformal manner. Hence, without loss of generality, we confine our attention to the functions that are univalent and analytic in the open unit disc $\{z/|z|<1\}$.

Notation: We denote by $\mathbf{A}$ the class of functions $\mathbf{f}(z)$ that are analytic in the open unit disc $\{z/|z|<1\}$ with the conditions $\mathbf{f}(0)=0,\ \mathbf{f}'(0)=1$. Then we say that

$$\mathbf{f}(z)\in\mathbf{A} \quad\Leftrightarrow\quad \mathbf{f}(z)=\textstyle\sum_{k=0}^{\infty}a_k z^k, |z|<1 \text{ with } a_0=0,\ a_1=1 \quad\Leftrightarrow\quad \mathbf{f}(z)=z+\textstyle\sum_{k=2}^{\infty}a_k z^k, |z|<1.$$

We denote by $\mathbf{U}$ the class of functions $\mathbf{f}(z)\in\mathbf{A}$ and are univalent in an open disc $\{z/|z|<c\leq 1\}$.

BEIRBARBACH Conjecture: In 1916, BEIRBERBACH proved that $|a_2|\leq 2$ for every $\mathbf{f}(z)$ in $\mathbf{U}$ whose Taylor's expansion about the origin is $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ .He also showed that $|a_2|=2$ for the function $\mathbf{f}(z)=z(1-\kappa z)^{-2},\ |\kappa|=1$, which is known as KOEBE's function. Note that singularity of KOEBE's function is $z=\kappa^{-1}$ which is outside the open unit disc $\{z/|z|<1\}$ since $|z|=\left|\kappa^{-1}\right|=|\kappa|^{-1}=1$; thus KOEBE's function is analytic in the open unit disc $\{z/|z|<1\}$. And $\mathbf{f}'(z)=z(-2)(1-\kappa z)^{-3}(-\kappa)+1(-2)(1-\kappa z)^{-2}$ implies $\mathbf{f}'(0)=0(-2)(1-\kappa 0)^{-3}(-\kappa)+1(1-\kappa 0)^{-2}=1.$

Clearly $\mathbf{f}(0)=0.$ So KOEBE's function is in $\mathbf{U}$.

Motivated by the extremal property of the KOEBE's function, BEIRBERBACH conjectured that $|a_n| \le n$ $(n = 2,3,4,\cdots)$ for every $\mathbf{f}(z) \in \mathbf{U}$. This is known as BEIRBERBACH conjecture which is a challenging problem in mathematics that took almost 70 years to prove it. LOUIS BRANZES has proved the conjecture in full in 1985.

Problem: $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k$ is univalent $\Leftrightarrow$ $c\mathbf{f}(z)$ is univalent in $\{z/|z| < c \le 1\}$.

Proof: Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ $\Rightarrow$ $c\mathbf{f}(z) = cz + \sum_{k=2}^{\infty} a_k c z^k = \mathbf{g}(z)$ where $0 < c < 1$.

$\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univalent in the disc $\{z/|z| < 1\}$

$\Leftrightarrow$ $\mathbf{f}(z_1) \ne \mathbf{f}(z_2)$ $\forall \{z_1, z_2\} \subset \{z/|z| < 1\}$ with $z_1 \ne z_2$

$\Leftrightarrow$ $c\mathbf{f}(z_1) \ne c\mathbf{f}(z_2)$ $\forall \{z_1, z_2\} \subset \{z/|z| < 1\}$ with $z_1 \ne z_2$

$\Leftrightarrow$ $\mathbf{g}(z_1) \ne \mathbf{g}(z_2)$ $\forall \{z_1, z_2\} \subset \{z/|z| < 1\}$ with $z_1 \ne z_2$

$\Leftrightarrow$ $c\mathbf{f}(z) = cz + \sum_{k=2}^{\infty} a_k c z^k = \mathbf{g}(z)$ is univalent in the disc $\{z/|z| < 1\}$. //

Theorem 1: If $\mathbf{f}(z) \in \mathbf{U}$, then $|a_k| \le 1$ $(k = 2,3,4,\cdots)$ where a_k is coefficient of z^k in taylor series of $\mathbf{f}(z)$.

Proof: Let $\mathbf{f}(z) \in \mathbf{U}$ $\Rightarrow$ $|a_k| \le k$ $(k = 2,3,4,\cdots)$ by BEIRBERBACH conjecture.

And $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Put $\mathbf{g}(z) = z\mathbf{f}'(z) = z[1 + \sum_{k=2}^{\infty} k a_k z^{k-1}] = z + \sum_{k=2}^{\infty} k a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k$ where $b_k = k a_k$

Let $z_1 \ne z_2$. Then $z_1^k \ne z_2^k$. Then $a_k z_1^k \ne a_k z_2^k$ $(k = 2,3,4,\cdots)$. But we say that the inequality $z_1 + \sum_{k=2}^{n} a_k z_1^k \ne z_2 + \sum_{k=2}^{n} a_k z_2^k$ may or may not hold. So we can do some work.

Since $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0$ $\Rightarrow$ $|z_1 - z_2| \ne 0$ $\Rightarrow$ $0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1$ $\Rightarrow$ $r - \rho = |z_2| - |z_1| < |z_2 - z_1|$ by triangle inequality.

Consider $\mathbf{g}(z_1) - \mathbf{g}(z_2) = z_1\mathbf{f}'(z_1) - z_2\mathbf{f}'(z_2) = [z_1 + \sum_{k=2}^{\infty} k a_k z_1^k] - [z_2 + \sum_{k=2}^{\infty} k a_k z_2^k]$

i.e. $\mathbf{g}(z_1) - \mathbf{g}(z_2) = z_1 - z_2 + [\sum_{k=2}^{\infty} k a_k z_1^k - \sum_{k=2}^{\infty} k a_k z_2^k] = z_1 - z_2 + \sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]$

By triangle inequality, $|z_1 - z_2 + \sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \ge |z_1 - z_2| - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$

$\Rightarrow$ $|\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge |z_1 - z_2| - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| > r - \rho - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$

Again by triangle inequality, we have

$$|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le \sum_{k=2}^{n} |k a_k [z_1^k - z_2^k]| = \sum_{k=2}^{\infty} k|a_k||z_1^k - z_2^k| \le \sum_{k=2}^{\infty} kk\,[|z_1^k| + |z_2^k|]$$

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le \sum_{k=2}^{\infty} k^2 [|z_1|^k + |z_2|^k] \le \sum_{k=2}^{\infty} k^2 [r^k + r^k] = \sum_{k=2}^{\infty} k^2\, 2r^k$.

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le 2\sum_{k=2}^{\infty} k^2 r^k = 2\sum_{k=2}^{\infty} [k(k-1) + k] r^k$

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le 2[\sum_{k=2}^{\infty} k(k-1) r^k + \sum_{k=2}^{\infty} k\, r^k]$

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le 2[r^2 \sum_{k=2}^{\infty} k(k-1) r^{k-2} + r\sum_{k=2}^{\infty} k\, r^{k-1}]$

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \le 2r^2 \sum_{k=2}^{\infty} \frac{d^2}{dr^2} r^k + 2r \sum_{k=2}^{\infty} \frac{d}{dr} r^k = 2r^2 \frac{d^2}{dr^2} \sum_{k=2}^{\infty} r^k + 2r \frac{d}{dr} \sum_{k=2}^{\infty} r^k$

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \le 2r^2 \frac{d^2}{dr^2}\left[-1 - r + \sum_{k=0}^{\infty} r^k\right] + 2r \frac{d}{dr}\left[-1 - r + \sum_{k=0}^{\infty} r^k\right]$

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \le 2r^2 \frac{d^2}{dr^2}\left[-1 - r + \frac{1}{1-r}\right] + 2r \frac{d}{dr}\left[-1 - r + \frac{1}{1-r}\right]$

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \le 2r^2\left[0 + \frac{2}{(1-r)^3}\right] + 2r\left[-1 + \frac{1}{(1-r)^2}\right] = 2r\left[\frac{2r}{(1-r)^3} - 1 + \frac{1}{(1-r)^2}\right]$

i.e. $\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \le 2r\left[\frac{2r+1-r}{(1-r)^3} - 1\right] = 2r\left[\frac{r+1}{(1-r)^3} - 1\right]$

$\Rightarrow \quad -\left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \ge -2r\left[\frac{r+1}{(1-r)^3} - 1\right] = 2r\left[1 - \frac{r+1}{(1-r)^3}\right]$

Thus we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > r - \rho - \left|\sum_{k=2}^{\infty} k a_k \left[z_1^k - z_2^k \right]\right| \ge r - \rho + 2r\left[1 - \frac{r+1}{(1-r)^3}\right] = r\left[3 - \frac{2(r+1)}{(1-r)^3}\right] - \rho$$

By taking limit $\rho = |z_1| \to 0$ on both sides since either $0 = |z_1| < |z_2|$ or $0 < |z_1| \le |z_2|$,

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > r\left[3 - \frac{2(r+1)}{(1-r)^3}\right] - \lim_{\rho \to 0} \rho = r\left[3 - \frac{2(r+1)}{(1-r)^3}\right] - 0 = r\left[3 - \frac{2(r+1)}{(1-r)^3}\right].$$

Hence, for $|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > 0$ *i.e.* $\mathbf{g}(z_1) \ne \mathbf{g}(z_2)$, we must have

$$3 - \frac{2r+2}{(1-r)^3} > 0 \quad \Leftrightarrow \quad 3 > \frac{2r+2}{(1-r)^3}$$

$\Leftrightarrow \quad 3(1-r)^3 > 2r + 2 \quad (\because r < 1 \;\; i.e. \;\; 0 < 1 - r) \quad \Leftrightarrow \quad 3\left[1 - r^3 - 3r(1-r)\right] > 2r + 2$

$\Leftrightarrow \quad 3 - 3r^3 - 9r + 9r^2 > 2 + 2r \quad \Leftrightarrow \quad 1 - 11r + 9r^2 - 3r^3 > 0$

$\Rightarrow \quad 1 - 11\frac{1}{R} + 9\frac{1}{R^2} - 3\frac{1}{R^3} > 0 \quad (rR = 1, \text{ where } R > 1 \text{ since } r < 1)$

$\Rightarrow \quad R^3 - 11R^2 + 9R - 3 > 0 \quad \Rightarrow \quad R^3 - 11R^2 + 9R > 3 \quad \Rightarrow \quad R(R^2 - 11R + 9) > 3$

$\Rightarrow \quad R\left[R - \frac{-(-11) - \sqrt{(-11)^2 - 4(1)9}}{2}\right]\left[R - \frac{-(-11) + \sqrt{(-11)^2 - 4(1)9}}{2}\right] > 3$

$\Rightarrow \quad R\left[R - \frac{11 - \sqrt{121 - 36}}{2}\right]\left[R - \frac{11 + \sqrt{121 - 36}}{2}\right] > 3$

$\Rightarrow \quad R\left[R - \frac{11 - \sqrt{121 - 36}}{2}\right]\left[R - \frac{11 + \sqrt{121 - 36}}{2}\right] > 3$

$\Rightarrow \quad R\left[R - \frac{11 - \sqrt{85}}{2}\right]\left[R - \frac{11 + \sqrt{85}}{2}\right] > 3 > 0 \quad (\sqrt{85} \cong 9.2195)$

$$\Rightarrow \quad \begin{cases} \textit{Either} \quad R < \frac{11 - \sqrt{85}}{2} \cong \frac{11 - 9.2195}{2} = \frac{2.2195}{2} = 1.10975, \\ \textit{or} \quad R > \frac{11 + \sqrt{85}}{2} \cong \frac{11 + 9.2195}{2} = \frac{20.2195}{2} = 10.10975 \end{cases}$$

Suppose that

$$1 < R < \frac{11-\sqrt{85}}{2} < \frac{11+\sqrt{85}}{2} \qquad i.e. \qquad 1 < R < 1.10975 < 10.10975$$

$$\Rightarrow \quad 0 < 1.10975 - R < 1.10975 - 1 = 0.10975, \qquad 0 < 10.10975 - R < 10.10975 - 1 = 9.10975$$

$$\Rightarrow \quad 0 < [1.10975 - R][10.10975 - R] < (0.10975)(9.10975) = 0.999795$$

$$\Rightarrow \quad 0 < R[R - 1.10975][R - 10.10975] < 1.10975(0.999795) = 1.10952257 < 3$$

This is contradiction. Thus our supposition is wrong. Hence we have

$$R > \frac{11+\sqrt{85}}{2} \quad \Rightarrow \quad \frac{1}{r} > \frac{11+\sqrt{85}}{2} \quad \Rightarrow \quad r < \frac{2}{11+\sqrt{85}} = \frac{2(11-\sqrt{85})}{121-85} = \frac{2(11-\sqrt{85})}{36}$$

$$i.e. \quad 0 \le \rho = |z_1| \le |z_2| = r < \frac{11-\sqrt{85}}{18} \cong 0.0989 < 1.$$

Thus, for $0 \le \rho = |z_1| \le |z_2| = r < c < 1$, we have

$$\left(c = \frac{11-\sqrt{85}}{18} \cong 0.0989 < 1\right), \qquad |\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge r\left[3 - \frac{2(1+r)}{(1-r)^3}\right] - \rho > 0$$

$i.e.$ $\mathbf{g}(z_1) \ne \mathbf{g}(z_2)$ $i.e.$ $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is univalent in $\{z / |z| < c < 1\}$.

$\Rightarrow$ $|b_k| \le k$ $(k = 2,3,4,\cdots)$ by BEIRBERBACH conjecture

$\Rightarrow$ $|k a_k| \le k$ or $k|a_k| \le k$ $(k = 2,3,4,\cdots)$ $\Rightarrow$ $|a_k| \le 1$ $(k = 2,3,4,\cdots).//$

Theorem 2: $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $|z| < 1$ is univalent in $\{z / |z| < 3^{-1} < 1\}$.

Proof: Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univalent in the open disc $\{z / |z| < c < 1\}$.

$\Rightarrow$ $|a_k| \le 1$ $(k = 2,3,4,\cdots)$ by **Theorem1.**

Let $z_1 \ne z_2 \Rightarrow z_1 - z_2 \ne 0 \Rightarrow |z_1 - z_2| \ne 0 \Rightarrow 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1 \Rightarrow r - \rho = |z_2| - |z_1| < |z_1 - z_2|$ by triangle inequality.

Consider

$$|\mathbf{f}(z_1) - \mathbf{f}(z_2)| = \left|\left[z_1 + \sum_{k=2}^{\infty} a_k z_1^k\right] - \left[z_2 + \sum_{k=2}^{\infty} a_k z_2^k\right]\right| = \left|z_1 - z_2 + \sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right|.$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right|$

$$\Rightarrow \quad |\mathbf{f}(z_1) - \mathbf{f}(z_2)| \ge |z_1 - z_2| - \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| > r - \rho - \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right|$$

Again by triangle inequality, we have

$$\left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{n} \left|a_k\left[z_1^k - z_2^k\right]\right| = \sum_{k=2}^{\infty} |a_k|\,|z_1^k - z_2^k| \le \sum_{k=2}^{\infty} 1\left[|z_1^k| + |z_2^k|\right]$$

$$i.e. \quad \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \le \sum_{k=2}^{\infty}\left[|z_1|^k + |z_2|^k\right] \le \sum_{k=2}^{\infty}\left[r^k + r^k\right] = \sum_{k=2}^{\infty} 2r^k = 2\sum_{k=2}^{\infty} r^k$$

$$i.e. \quad \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \le 2\left[\sum_{k=0}^{\infty} r^k - 1 - r\right] = 2\left[\frac{1}{1-r} - 1 - r\right] = 2\left[\frac{1-(1-r)}{1-r} - r\right]$$

$$i.e. \quad \left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \le 2\left[\frac{r}{1-r} - r\right] = 2r\left[\frac{1}{1-r} - 1\right]$$

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Thus we have

$$|\mathbf{f}(z_1)-\mathbf{f}(z_2)| > r-\rho-\left|\sum_{k=2}^{\infty} a_k\left[z_1^k - z_2^k\right]\right| \geq r-\rho+2r\left[1-\frac{1}{1-r}\right] = r\left[3-\frac{2}{1-r}\right]-\rho$$

By taking limit $\rho = |z_1| \to 0$ on both sides since either $0 = |z_1| < |z_2|$ or $0 < |z_1| \leq |z_2|$,

$$|\mathbf{f}(z_1)-\mathbf{f}(z_2)| > r\left[3-\frac{2}{1-r}\right] - \lim_{\rho\to 0}\rho = r\left[3-\frac{2}{1-r}\right] - 0 = r\left[3-\frac{2}{1-r}\right].$$

Hence, for $|\mathbf{f}(z_1)-\mathbf{f}(z_2)| > 0$ *i.e.* $\mathbf{f}(z_1) \neq \mathbf{f}(z_2)$, we must have

$$3-\frac{2}{1-r} > 0 \quad \Leftrightarrow \quad 3 > \frac{2}{1-r} \quad \Leftrightarrow \quad 3(1-r) > 2 \quad (\because r<1 \quad i.e. \quad 0<1-r)$$

$$\Leftrightarrow \quad 3-3r > 2 \quad \Leftrightarrow \quad 1 < 3r \quad \Leftrightarrow \quad 3r < 1 \quad \Leftrightarrow \quad r < 3^{-1}$$

Therefore $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univalent in the open disc $\{z/|z| < 3^{-1} < 1\}$.//

Definition 2: A function $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $|z| < 1$ is said to be *cap like function* in the open disc $\{z/|z| < c \leq 1\}$ if

$$\mathbf{Re}\left[1+\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] > 0, \quad |z| < c.$$

Definition 3: A function $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $|z| < 1$ is said to be *star like function* in the open disc $\{z/|z| < c \leq 1\}$ if $\mathbf{f}(z)$ univalent in $\{z/|z| < c \leq 1\}$, and

$$\mathbf{Re}\left[\frac{z\,\mathbf{f}'(z)}{\mathbf{f}(z)}\right] > 0, \quad |z| < c.$$

Theorem 3: If $\mathbf{f}(z) \in \mathbf{U}$, then $\mathbf{f}(z)$ is cap like function in $\{z/|z| < 6^{-1} < 1\}$.

Proof: Let $\mathbf{f}(z) \in \mathbf{U}$. Then $|a_k| \leq 1$ $(k = 2,3,4,\cdots)$ by **Theorem1;**

And $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univalent in $\{z/|z| < 3^{-1}\}$ by **Theorem2.** Let us consider

$$\mathbf{Re}\left[1+\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] = 1 + \mathbf{Re}\left[\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right].$$

We know that

$$\left[\mathbf{Re}\left[\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right]\right]^2 \leq \left|\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right|^2 \quad \Rightarrow \quad -\left|\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right| \leq \mathbf{Re}\left[\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] \leq \left|\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right|$$

$$\Rightarrow \quad 1-\left|\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right| \leq 1+\mathbf{Re}\left[\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] = \mathbf{Re}\left[1+\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right]$$

$$\Rightarrow \quad \mathbf{Re}\left[1+\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] \geq 1-\left|\frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)}\right| = 1-\frac{|z\,\mathbf{f}''(z)|}{|\mathbf{f}'(z)|} = \frac{|\mathbf{f}'(z)|-|z\,\mathbf{f}''(z)|}{|\mathbf{f}'(z)|}$$

We have $\mathbf{f}'(z) = 1 + \sum_{k=2}^{\infty} a_k k z^{k-1} \quad \Rightarrow \quad \mathbf{f}''(z) = \sum_{k=2}^{\infty} a_k k(k-1) z^{k-2}$

$$\Rightarrow \quad |z\,\mathbf{f}''(z)| = \left|\sum_{k=2}^{\infty} a_k k(k-1) z^{k-1}\right| \leq \sum_{k=2}^{\infty} |a_k|\, k(k-1)\,|z|^{k-1} \leq \sum_{k=2}^{\infty} 1k(k-1)\,|z|^{k-1}$$

By Triangle inequality, $|\mathbf{f}'(z)| = |1+\sum_{k=2}^{\infty} a_k k\, z^{k-1}| \geq 1 - |\sum_{k=2}^{\infty} a_k k\, z^{k-1}|$

But $|\sum_{k=2}^{\infty} a_k k\, z^{k-1}| \leq \sum_{k=2}^{\infty} |a_k|\, k\, |z|^{k-1} \leq \sum_{k=2}^{\infty} 1k\, |z|^{k-1} = \sum_{k=2}^{\infty} k |z|^{k-1}$

Put $|z| = r \quad \Rightarrow \quad |z\mathbf{f}''(z)| \leq \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1}$, and $|\mathbf{f}'(z)| \geq 1 - \sum_{k=2}^{\infty} k\, r^{k-1}$

$$\Rightarrow \quad |\mathbf{f}'(z)| - |z\mathbf{f}''(z)| \geq 1 - \sum_{k=2}^{\infty} k\, r^{k-1} - \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1} = 1 - \sum_{k=2}^{\infty} k^2 r^{k-1}.$$

We have

$$\sum_{k=2}^{\infty} k^2 r^{k-1} = \sum_{k=2}^{\infty}[k(k-1)+k] r^{k-1} = \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1}$$

$$= \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1} = r\sum_{k=2}^{\infty} k(k-1) r^{k-2} + \sum_{k=2}^{\infty} k\, r^{k-1}.$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = r\sum_{k=2}^{\infty}\frac{d^2}{dr^2} r^k + \sum_{k=2}^{\infty}\frac{d}{dr} r^k = r\frac{d^2}{dr^2}\sum_{k=2}^{\infty} r^k + \frac{d}{dr}\sum_{k=2}^{\infty} r^k = \left[r\frac{d^2}{dr^2}+\frac{d}{dr}\right]\sum_{k=2}^{\infty} r^k$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = \left[r\frac{d^2}{dr^2}+\frac{d}{dr}\right]\left[\sum_{k=0}^{\infty} r^k - 1 - r\right] = \left[r\frac{d^2}{dr^2}+\frac{d}{dr}\right]\left[\frac{1}{1-r} - 1 - r\right]$$

i.e. $$\sum_{k=2}^{\infty} k^2 r^{k-1} = r\frac{2}{(1-r)^3} + \frac{1}{(1-r)^2} - 1 = \frac{2r+1-r}{(1-r)^3} - 1 = \frac{r+1}{(1-r)^3} - 1$$

Thus we have

$$|\mathbf{f}'(z)| - |z\mathbf{f}''(z)| \geq 1 - \sum_{k=2}^{\infty} k^2 r^{k-1} = 1 - \frac{r+1}{(1-r)^3} + 1 = 2 - \frac{r+1}{(1-r)^3}$$

Hence, for $|\mathbf{f}'(z)| - |z\mathbf{f}''(z)| > 0$, we must have

$$|\mathbf{f}'(z)| - |z\mathbf{f}''(z)| \geq 2 - \frac{r+1}{(1-r)^3} > 0 \quad \Leftrightarrow \quad 2 > \frac{r+1}{(1-r)^3} \quad \Leftrightarrow \quad 2(1-r)^3 > r+1$$

$$\Leftrightarrow \quad 2[1-r^3-3r(1-r)] > r+1 \quad \Leftrightarrow \quad 2-2r^3-6r+6r^2 > r+1$$

$$\Leftrightarrow \quad 1-7r+6r^2-2r^3 > 0 \quad \Leftrightarrow \quad 1-7\frac{1}{R}+6\frac{1}{R^2}-2\frac{1}{R^3} > 0 \quad (Rr=1, R>1 \text{ since } r<1)$$

$$\Leftrightarrow \quad R^3-7R^2+6R-2>0 \quad \Leftrightarrow \quad R^3-7R^2+6R>2 \quad \Leftrightarrow \quad R(R^2-7R+6)>2$$

$$\Leftrightarrow \quad R(R-1)(R-6)>2 \quad \Leftrightarrow \quad R>6 \quad (\because R>1).$$

$$\Leftrightarrow \quad \frac{1}{r}>6 \quad \Rightarrow \quad r<\frac{1}{6} \quad \Leftrightarrow \quad |z|<\frac{1}{6}$$

Thus, for $|z| < 6^{-1} < 1$, we have

$$|\mathbf{f}'(z)| - |z\mathbf{f}''(z)| \geq 2 - \frac{r+1}{(1-r)^3} > 0 \quad \Rightarrow \quad \mathbf{Re}\left[1+\frac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right] \geq \frac{|\mathbf{f}'(z)| - |z\mathbf{f}''(z)|}{|\mathbf{f}'(z)|} > 0.$$

Hence $\mathbf{f}(z)$ is cap like function in the open disc $|z| < 6^{-1} < 1$. //

Theorem 4: Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is cap like function and is univlent. Then $z\mathbf{f}'(z)$ is star like in $\{z / |z| < c \leq 1\}$ where

$$c = 1 - \frac{\sqrt{6}}{3} = \frac{3-\sqrt{6}}{3}.$$

Proof : Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univlent $\Rightarrow$ $|a_k| \le 1 \quad (k = 2, 3, 4, \cdots)$

Put $\mathbf{g}(z) = z\mathbf{f}'(z) = z[1 + \sum_{k=2}^{\infty} k a_k z^{k-1}] = z + \sum_{k=2}^{\infty} k a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k$ where $b_k = k a_k$

Let $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0 \quad \Rightarrow \quad |z_1 - z_2| \ne 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| \le |z_1 - z_2|$ by triangle inequality.

Consider $\mathbf{g}(z_1) - \mathbf{g}(z_2) = [z_1 + \sum_{k=2}^{\infty} k a_k z_1^k] - [z_2 + \sum_{k=2}^{\infty} k a_k z_2^k]$

i.e. $\mathbf{g}(z_1) - \mathbf{g}(z_2) = z_1 - z_2 + [\sum_{k=2}^{\infty} k a_k z_1^k - \sum_{k=2}^{\infty} k a_k z_2^k] = z_1 - z_2 + \sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]$

$\Rightarrow \quad |\mathbf{g}(z_1) - \mathbf{g}(z_2)| = |z_1 - z_2 + \sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$

By triangle inequality, $|z_1 - z_2 + \sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \ge |z_1 - z_2| - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$

$\Rightarrow \quad |\mathbf{g}(z_1) - \mathbf{g}(z_2)| \ge |z_1 - z_2| - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| > r - \rho - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$

Again by triangle inequality, we have

$$|\sum_{k=2}^{n} k a_k [z_1^k - z_2^k]| \le \sum_{k=2}^{n} |k a_k [z_1^k - z_2^k]| = \sum_{k=2}^{n} k |a_k| |z_1^k - z_2^k| \le \sum_{k=2}^{n} k.1.[|z_1^k| + |z_2^k|]$$

i.e. $|\sum_{k=2}^{n} k a_k [z_1^k - z_2^k]| \le \sum_{k=2}^{n} k[|z_1|^k + |z_2|^k] \le \sum_{k=2}^{n} k[r^k + r^k] = \sum_{k=2}^{\infty} k\, 2r^k = 2\sum_{k=2}^{\infty} k\, r^k$

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le 2r \sum_{k=2}^{\infty} k\, r^{k-1} = 2r \sum_{k=2}^{\infty} \frac{d}{dr} r^k = 2r \frac{d}{dr} \sum_{k=2}^{\infty} r^k = 2r \frac{d}{dr}\left[\sum_{k=0}^{\infty} r^k - 1 - r\right]$

i.e. $|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \le 2r \frac{d}{dr}\left[\frac{1}{1-r} - 1 - r\right] = 2r\left[\frac{1}{(1-r)^2} - 0 - 1\right] = 2r\left[\frac{1}{(1-r)^2} - 1\right]$

$$\Rightarrow \quad -|\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]| \ge 2r\left[1 - \frac{1}{(1-r)^2}\right]$$

Thus we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > r - \rho - |\sum_{k=2}^{\infty} k a_k [z_1^k - z_2^k]|$$

$$\ge r - \rho + 2r\left[1 - \frac{1}{(1-r)^2}\right] = r + 2r - \frac{2r}{(1-r)^2} - \rho = r\left[3 - \frac{2}{(1-r)^2}\right] - \rho$$

By taking limit $\rho = |z_1| \to 0$ on both sides since either $0 = |z_1| < |z_2|$ or $0 < |z_1| \le |z_2|$,

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > r\left[3 - \frac{2}{(1-r)^2}\right] - \lim_{\rho \to 0} \rho = r\left[3 - \frac{2}{(1-r)^2}\right] - 0 = r\left[3 - \frac{2}{(1-r)^2}\right].$$

Hence, for $|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > 0$, we must have

$$3 - \frac{2}{(1-r)^2} > 0 \quad \Leftrightarrow \quad 3 > \frac{2}{(1-r)^2} \quad \Leftrightarrow \quad (1-r)^2 > \frac{2}{3}$$

$$\Leftrightarrow \quad 1 - r > \frac{\sqrt{2}}{\sqrt{3}} \quad \Leftrightarrow \quad 1 - \frac{\sqrt{2}}{\sqrt{3}} > r \quad \Leftrightarrow \quad r < 1 - \frac{\sqrt{2}}{\sqrt{3}} = 1 - \frac{\sqrt{6}}{3} \cong 0.1835 < 1.$$

Thus, for $0 \le \rho = |z_1| \le |z_2| = r < c < 1$, we have

$$|\mathbf{g}(z_1) - \mathbf{g}(z_2)| > r\left[3 - \frac{2}{(1-r)^2}\right] - \rho > 0, \qquad \left(c = 1 - \frac{\sqrt{6}}{3} = \frac{3 - \sqrt{6}}{3} = 0.1835\right)$$

$\Rightarrow \quad \mathbf{g}(z_1) \ne \mathbf{g}(z_2)$ for $z_1 \ne z_2$ *i.e.* $\mathbf{g}(z)$ is univalent function in $\{z / |z| < 1 - 3^{-1}\sqrt{6} \le 1\}$.

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Since $\mathbf{f}(z)$ is cap like function, we have

$$0<\mathbf{Re}\left[1+\frac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right]=\mathbf{Re}\left[\frac{\mathbf{f}'(z)+z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right]=\mathbf{Re}\left[\frac{1}{\mathbf{f}'(z)}\frac{d}{dz}[z\mathbf{f}'(z)]\right]=\mathbf{Re}\left[\frac{1}{z\mathbf{f}'(z)}z\frac{d}{dz}[z\mathbf{f}'(z)]\right],$$

Now $\mathbf{g}'(z)=z\mathbf{f}''(z)+\mathbf{f}'(z)$ will exist since $\mathbf{f}(z)$ is analytic; and $\mathbf{g}(0)=0\mathbf{f}'(0)=0\times 1=0$

Thus $\mathbf{g}'(0)=0\mathbf{f}''(0)+\mathbf{f}'(0)=\mathbf{f}'(0)=1$ and $\mathbf{g}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$.

Hence $\mathbf{g}(z)=z\mathbf{f}'(z)$ is star like function is in the disc $|z|<c=1-3^{-1}\sqrt{6}<1.//$

Problem: Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ is star like function. Then

$$0<\mathbf{Re}\left[\frac{1}{z\mathbf{f}(z)}z\frac{d}{dz}[z\mathbf{f}(z)]\right]$$

and $z\mathbf{f}(z)$ is analytic in the open unit disc$\{z/|z|<1\}$, but $z\mathbf{f}(z)$ is not univalent in the open disc $\{z/|z|<r\le 1\}$, thus $z\mathbf{f}(z)$ is not star like function.

Proof: Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ is star like function. Then

$$0<\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]<\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]+1=\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+1\right]=\mathbf{Re}\left[\frac{z\mathbf{f}'(z)+\mathbf{f}(z)}{\mathbf{f}(z)}\right]=\mathbf{Re}\left[\frac{1}{\mathbf{f}(z)}\frac{d}{dz}[z\mathbf{f}(z)]\right]$$

And $\mathbf{f}(z)$ is univalent in an open disc $\{z/|z|<r<1\}$

And $\mathbf{g}(z)=z\mathbf{f}(z)=z[z+\sum_{k=2}^{\infty}a_k z^k]=z^2+\sum_{k=2}^{\infty}a_k z^{k+1}$ is analytic in open unit disc $\{z/|z|<1\}$

$\Rightarrow$ $\mathbf{g}(0)=0\mathbf{f}(0)=0\times 0=0,$ $\mathbf{g}'(z)=z\mathbf{f}'(z)+\mathbf{f}(z)$ but $\mathbf{g}'(0)=0\mathbf{f}'(0)+\mathbf{f}(0)=\mathbf{f}'(0)=0\neq 1.$

Consider

$$\mathbf{g}(z_1)-\mathbf{g}(z_2)=z_1\mathbf{f}(z_1)-z_2\mathbf{f}(z_2)=[z_1^2+\sum_{k=2}^{\infty}a_k z_1^{k+1}]-[z_2^2+\sum_{k=2}^{\infty}a_k z_2^{k+1}]$$
$$=z_1^2-z_2^2+\sum_{k=2}^{\infty}a_k z_1^{k+1}-\sum_{k=2}^{\infty}a_k z_2^{k+1}=z_1^2-z_2^2+\sum_{k=2}^{\infty}a_k[z_1^{k+1}-z_2^{k+1}]$$

Let $z_1\neq z_2$ in the open disc $\{z/|z|<r\le 1\}$ such that $z_1=-z_2$ $\Rightarrow$ $z_1^2=z_2^2$

$\Rightarrow$ $\mathbf{g}(z_1)-\mathbf{g}(z_2)=\sum_{k=2}^{\infty}a_k z_1^{k+1}-\sum_{k=2}^{\infty}a_k z_2^{k+1}$ may or may not be 0.

Thus $\mathbf{g}(z)$ is not univalent in $\{z/|z|<r\le 1\}$. So $\mathbf{g}(z)=z\mathbf{f}(z)$ is not star like function.//

Definition 4: A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z|<1\}$ with $\mathbf{f}(0)=0,$ $\mathbf{f}'(0)=1$ is said to be *cap like function of order* $\alpha(c)$ in the open disc $\{z/|z|<c\le 1$ if

$$\mathbf{Re}\left[1+\frac{z\mathbf{f}''(z)}{\mathbf{f}'(z)}\right]>\alpha(c),\quad |z|<c\le 1,\quad 0\le\alpha(c)<1.$$

Definition 5: A function $\mathbf{f}(z)$ that is analytic in the open unit disc $\{z/|z|<1\}$ and univalent in open disc $\{z/|z|<c\le 1\}$ with $\mathbf{f}(0)=0,$ $\mathbf{f}'(0)=1$ is said to be *star like function of order* $\alpha(c)$ if

$$\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]>\alpha(c),\quad |z|<c\le 1,\quad 0\le\alpha(c)<1.$$

Theorem 5: If $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ in $|z|<1$ is star like function of order $\alpha<1$, then $(z+1)\mathbf{f}(z)$ is star like function of order $\alpha\geq 0.4$.

Proof: Let $\mathbf{f}(z)$ is star like function of order $\alpha<1$.

$$\Rightarrow \quad \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]>\alpha, \qquad \mathbf{f}(z)\in\mathbf{U}$$

By **Theorem 1**, $|a_k|\leq 1 \quad (k=2,3,4,\cdots)$ since $\mathbf{f}(z)\in\mathbf{U}$.

Now $\mathbf{g}(z)=(z+1)\mathbf{f}(z)$ is analytic in the open unit disc $\{z/|z|<1\}$ since $\mathbf{f}(z)$ is analytic,

$$\Rightarrow \begin{cases} \mathbf{g}(0)=1\mathbf{f}(0)=1\times 0=0, \quad \text{and} \\ \mathbf{g}'(z)=(z+1)\mathbf{f}'(z)+\mathbf{f}(z) \quad \Rightarrow \quad \mathbf{g}'(0)=(0+1)\mathbf{f}'(0)+\mathbf{f}(0)=\mathbf{f}'(0)+0=1. \end{cases}$$

Consider $z\mathbf{f}(z)=z\left[z+\sum_{k=2}^{\infty}a_k z^k\right]=z^2+\sum_{k=2}^{\infty}a_k z^{k+1}$.

$$\Rightarrow \quad (z+1)\mathbf{f}(z)=z^2+\sum_{k=2}^{\infty}a_k z^{k+1}+z+\sum_{k=2}^{\infty}a_k z^k$$

$$=z+z^2+\sum_{k-1=2}^{\infty}a_{k-1}z^{k-1+1}+\sum_{k=2}^{\infty}a_k z^k$$

$$=z+z^2+\sum_{k=3}^{\infty}a_{k-1}z^k+a_2+\sum_{k=3}^{\infty}a_k z^k=z+(1+a_2)z^2+\sum_{k=3}^{\infty}(a_{k-1}+a_k)z^k$$

Put $1+a_2=b_2$, $a_{k-1}+a_k=b_k \quad (k=3,4,5,\cdots) \quad \Rightarrow \quad \mathbf{g}(z)=(z+1)\mathbf{f}(z)=z+\sum_{k=2}^{\infty}b_k z^k$.

Let $z_1\neq z_2$, we have $z_1-z_2\neq 0 \quad \Rightarrow \quad |z_1-z_2|\neq 0 \quad \Rightarrow \quad 0<|z_1-z_2|$.

Let $\rho=|z_1|\leq|z_2|=r<1 \quad \Rightarrow \quad 0\leq r-\rho=|z_2|-|z_1|<|z_1-z_2|$ by triangle inequality.

Consider

$$\mathbf{g}(z_1)-\mathbf{g}(z_2)=(z_1+1)\mathbf{f}'(z_1)-(z_2+1)\mathbf{f}'(z_2)$$

$$=\left[z_1+\sum_{k=2}^{\infty}b_k z_1^k\right]-\left[z_2+\sum_{k=2}^{\infty}b_k z_2^k\right]$$

$$=z_1-z_2+\left[\sum_{k=2}^{\infty}b_k z_1^k-\sum_{k=2}^{\infty}b_k z_2^k\right]=z_1-z_2+\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]$$

$$\Rightarrow \quad |\mathbf{g}(z_1)-\mathbf{g}(z_2)|=\left|z_1-z_2+\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|$$

By triangle inequality, $\left|z_1-z_2+\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|\geq|z_1-z_2|-\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|$

$$\Rightarrow \quad |\mathbf{g}(z_1)-\mathbf{g}(z_2)|\geq|z_1-z_2|-\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|>r-\rho-\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|$$

Consider

$$\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|\leq\sum_{k=2}^{\infty}|b_k|\left[|z_1|^k+|z_2|^k\right]<\sum_{k=2}^{\infty}|b_k|\left[r^k+r^k\right]=2\sum_{k=2}^{\infty}|b_k|r^k.$$

$$|b_2|=|1+a_2|\leq 1+|a_2|\leq 1+1=2, \qquad |b_k|=|a_{k-1}+a_k|\leq|a_{k-1}|+|a_k|\leq 1+1=2.$$

Then we have

$$\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|\leq 2\sum_{k=2}^{\infty}|b_k|r^k\leq 2\sum_{k=2}^{\infty}2r^k=4\sum_{k=2}^{\infty}r^k=4\left[\sum_{k=0}^{\infty}r^k-1-r\right]$$

Thus we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)|\geq|z_1-z_2|-\left|\sum_{k=2}^{\infty}b_k\left[z_1^k-z_2^k\right]\right|>r-\rho-4\sum_{k=2}^{\infty}r^k$$

i.e. $$|\mathbf{g}(z_1)-\mathbf{g}(z_2)|>r-\rho-4\frac{r^2}{1-r}=r\left[1-\frac{4r}{1-r}\right]-\rho$$

By taking limit $\rho=|z_1|\to 0$ on both sides since either $0=|z_1|<|z_2|$ or $0<|z_1|\leq|z_2|$,

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > r\left[1-\frac{4r}{1-r}\right]-\mathbf{lim}_{\rho\to 0}\,\rho = r\left[1-\frac{4r}{1-r}\right]-0 = r\left[1-\frac{4r}{1-r}\right].$$

Hence, for $|\mathbf{g}(z_1)-\mathbf{g}(z_2)|>0$, we must have

$$1-\frac{4r}{1-r}>0 \quad\Leftrightarrow\quad 1>\frac{4r}{1-r} \quad\Leftrightarrow\quad 1-r>4r \quad (\because 0<r<1 \quad i.e.\quad 0<1-r)$$

$$\Leftrightarrow \quad 1>5r \quad\Leftrightarrow\quad 5r<1 \quad\Leftrightarrow\quad r<0.2$$

Thus, for $0\le\rho=|z_1|\le|z_2|=r<0.2,$ we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > r\left[1-\frac{4r}{1-r}\right]-\rho>0 \qquad i.e. \qquad \mathbf{g}(z_1)\ne\mathbf{g}(z_2)$$

Thus $\mathbf{g}(z)$ is univalent in the open disc $\{z/|z|<0.2<1\}$.

We have

$$\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)] = \frac{z[(z+1)\mathbf{f}'(z)+\mathbf{f}(z)]}{(z+1)\mathbf{f}(z)} = \frac{(z+1)z\mathbf{f}'(z)+z\mathbf{f}(z)}{(z+1)\mathbf{f}(z)} = \frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+\frac{z}{z+1}$$

Consider

$$\mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}+\frac{z}{z+1}\right] = \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right]+\mathbf{Re}\left[\frac{z}{z+1}\right] > \alpha+\mathbf{Re}\left[\frac{z}{z+1}\right] = \alpha+\mathbf{Re}\left[\frac{z+1-1}{z+1}\right]$$

$$i.e. \qquad \mathbf{Re}\left[\frac{z}{(z+1)\mathbf{f}(z)}\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > \alpha+\mathbf{Re}\left[1-\frac{1}{z+1}\right] = \alpha+1-\mathbf{Re}\left[\frac{1}{z+1}\right]$$

$$i.e. \qquad \mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > \alpha+1-\mathbf{Re}\left[\frac{1}{x+\iota y+1}\right] = \alpha+1-\mathbf{Re}\left[\frac{1+x-\iota y}{(1+x)^2+y^2}\right]$$

$$i.e. \qquad \mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > \alpha+1-\frac{1+x}{(1+x)^2+y^2}$$

Let $|z|<c<1 \quad\Rightarrow\quad |z|^2<c^2<1 \quad\Rightarrow\quad x^2+y^2<c^2<1$

$$\Rightarrow \quad x^2<x^2+y^2<c^2<1 \quad\Rightarrow\quad -1<-c<x<c<1 \quad\Rightarrow\quad 0<1-c<1+x<1+c<2$$

But $\quad 0<(1+x)^2<(1+x)^2+y^2$

$$\Rightarrow \quad \frac{1}{(1+x)^2+y^2}<\frac{1}{(1+x)^2} \quad\Rightarrow\quad \frac{1+x}{(1+x)^2+y^2}<\frac{1+x}{(1+x)^2}=\frac{1}{1+x}<\frac{1}{1-c}$$

$$\Rightarrow \quad -\frac{1+x}{(1+x)^2+y^2}>-\frac{1}{1-c}$$

$$\Rightarrow \quad \alpha+1-\frac{1+x}{(1+x)^2+y^2}>\alpha+1-\frac{1}{1-c}=\frac{\alpha+1-(\alpha+1)c-1}{1-c}=\frac{\alpha-2c}{1-c}$$

Thus we have

$$\mathbf{Re}\left[\frac{1}{(z+1)\mathbf{f}(z)}z\frac{d}{dz}[(z+1)\mathbf{f}(z)]\right] > \alpha+1-\frac{1+x}{(1+x)^2+y^2} > \frac{\alpha-2c}{1-c}\ge 0$$

for $\alpha-2c\ge 0$ or $\alpha\ge 2c$ or $2c\le\alpha$ or $c\le 0.5\alpha<1$ since $\alpha<1$.

$$c=0.2\le 0.5\alpha<1 \quad\Rightarrow\quad \frac{2}{5}\le\alpha \quad\Rightarrow\quad \frac{\alpha-2c}{1-c}=\frac{\alpha-2(0.2)}{1-0.2}=\frac{\alpha-0.4}{0.8}\ge 0$$

So $\mathbf{g}(z)=(z+1)\mathbf{f}(z)$ is star like function of order $\alpha\ge 0.4$ in the open disc $|z|<0.2<1.//$

Definition 6 : n-th partial sum of the function $\mathbf{f}(z)=\sum_{k=0}^{\infty}a_k z^k$ is $s_n(z,\mathbf{f})=\sum_{k=0}^{n}a_k z^k$.

Theorem 6 : Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ in $\{z/|z|<1\}$ is univalent. Then $s_n(z,\mathbf{f})$ is analytic in the disc $\{z/|z|<1\}$ with $s_n(0,\mathbf{f})=0,\ s_n'(0,\mathbf{f})=1,$ and is univalent function in $|z|<3^{-1}<1$ for all integers $n=2,3,4\cdots$.

Proof : $s_n(z,\mathbf{f})=z+\sum_{k=2}^{n}a_k z^k$ in $|z|<1$ is analytic with $s_n(0,\mathbf{f})=0,\ s_n'(0,\mathbf{f})=1$.

Let $z_1\neq z_2$, we have $z_1-z_2\neq 0 \Rightarrow |z_1-z_2|\neq 0 \Rightarrow 0<|z_1-z_2|$.

Let $\rho=|z_1|\le|z_2|=r<1 \Rightarrow 0\le r-\rho=|z_2|-|z_1|<|z_1-z_2|$ by triangle inequality.

Consider

$$s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})=\left[z_1+\sum_{k=2}^{n}a_k z_1^k\right]-\left[z_2+\sum_{k=2}^{n}a_k z_2^k\right]$$

$$=z_1-z_2+\left[\sum_{k=2}^{n}a_k z_1^k-\sum_{k=2}^{n}a_k z_2^k\right]=z_1-z_2+\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]$$

$$\Rightarrow\quad |s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})|=\left|z_1-z_2+\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|$$

By triangle inequality, $\left|z_1-z_2+\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\ge|z_1-z_2|-\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|$

$$\Rightarrow\quad |s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})|\ge|z_1-z_2|-\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|>r-\rho-\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|$$

Note that $\mathbf{f}(z)\in\mathbf{U} \Rightarrow |a_k|\le 1\ \ (k=2,3,4,\cdots)$ by **Theorem 1.**

Again by triangle inequality,

$$\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\le\sum_{k=2}^{n}\left|a_k\left[z_1^k-z_2^k\right]\right|=\sum_{k=2}^{n}|a_k||z_1^k-z_2^k|\le\sum_{k=2}^{n}1\left[|z_1^k|+|z_2^k|\right]$$

i.e. $\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\le\sum_{k=2}^{n}\left[|z_1|^k+|z_2|^k\right]\le\sum_{k=2}^{n}\left[r^k+r^k\right]$ since $|z_1|\le|z_2|=r$.

i.e. $\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\le\sum_{k=2}^{n}2\,r^k=2\sum_{k=2}^{n}r^k=2\left[\sum_{k=0}^{n}r^k-1-r\right]=2\left[\dfrac{1-r^{n+1}}{1-r}-1-r\right]$

$$\Rightarrow\quad -\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\ge-2\left[\frac{1-r^{n+1}}{1-r}-1-r\right]=2\left[1+r-\frac{1-r^{n+1}}{1-r}\right]=2\left[\frac{1-r^2-1+r^{n+1}}{1-r}\right]$$

i.e. $-\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\ge 2\left[\dfrac{-r^2+r^{n+1}}{1-r}\right]=\dfrac{-2r^2}{1-r}+\dfrac{2r^{n+1}}{1-r}\ge\dfrac{-2r^2}{1-r}+0=\dfrac{-2r^2}{1-r}$

Thus we have

$$|s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})|>r-\rho-\left|\sum_{k=2}^{n}a_k\left[z_1^k-z_2^k\right]\right|\ge r-\rho-\frac{2r^2}{1-r}=r\left[1-\frac{2r}{1-r}\right]-\rho.$$

By taking limit $\rho=|z_1|\to 0$ on both sides since either $0=|z_1|<|z_2|$ or $0<|z_1|\le|z_2|$,

$$|s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})|>r\left[1-\frac{2r}{1-r}\right]-\lim_{\rho\to0}\rho=r\left[1-\frac{2r}{1-r}\right]-0=r\left[1-\frac{2r}{1-r}\right].$$

Hence, for $|s_n(z_1,\mathbf{f})-s_n(z_2,\mathbf{f})|>0$, we must have

$$1-\frac{2r}{1-r}>0 \iff 1>\frac{2r}{1-r} \iff 1-r>2r \quad (\because 0<r<1\ \ i.e.\ \ 0<1-r)$$

$$\iff 1>5r \iff 3r<1 \iff r<3^{-1}$$

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Thus, for $0 \le \rho = |z_1| \le |z_2| = r < 3^{-1}$, we have

$$|s_n(z_1,\mathbf{f}) - s_n(z_2,\mathbf{f}| > r\left[1 - \frac{2r}{1-r}\right] - \rho > 0 \qquad i.e. \qquad s_n(z_1,\mathbf{f}) \ne s_n(z_2,\mathbf{f})$$

Hence $s_n(z,\mathbf{f})$ is univalent function in the disc $|z| < 3^{-1}$ for all $n.//$

Theorem 7 : If $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is star like function in $\{z/|z|<1\}$, then $s_n(z,\mathbf{f})$ are star like functions in $|z| < 1 - \sqrt[-4]{2}$.

Proof : Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is star like function in $\{z/|z|<1\}$.

$$\Rightarrow \qquad \mathbf{Re}\left[\frac{z\mathbf{f}'(z)}{\mathbf{f}(z)}\right] > 0, \quad |z| < 1.$$

And $\mathbf{f}(z)$ is univalent $\quad\Rightarrow\quad |a_k| \le 1, \quad k = 2,3,4,\cdots$ by Theorem**1.**

Partial sum of $\mathbf{f}(z)$ is $s_n(z,\mathbf{f}) = z + \sum_{k=2}^{n} a_k z^k$. Consider

$$\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})} = \frac{z\left[1 + \sum_{k=2}^{n} a_k k\, z^{k-1}\right]}{z + \sum_{k=2}^{n} a_k z^k} = \frac{1 + \sum_{k=2}^{n} a_k k\, z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} = \frac{1 + \sum_{k=2}^{n} a_k k\, z^{k-1}}{\mathbf{h}_n(z)}$$

$$\Rightarrow \qquad \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] = \mathbf{Re}\left[\frac{1 + \sum_{k=2}^{n} a_k k\, z^{k-1}}{\mathbf{h}_n(z)}\right] = \mathbf{Re}\left[\frac{\overline{\mathbf{h}_n(z)}\left[1 + \sum_{k=2}^{n} a_k k\, z^{k-1}\right]}{\overline{\mathbf{h}_n(z)}\,\mathbf{h}_n(z)}\right]$$

$$i.e. \qquad \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] = \mathbf{Re}\left[\frac{\overline{\mathbf{h}_n(z)} + \overline{\mathbf{h}_n(z)}\sum_{k=2}^{n} a_k k\, z^{k-1}}{|\mathbf{h}_n(z)|^2}\right]$$

$$i.e. \qquad \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] = \frac{1}{|\mathbf{h}_n(z)|^2}\mathbf{Re}\left[1 + \sum_{k=2}^{n} \bar{a}_k\, \bar{z}^{k-1} + \left[1 + \sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1}\right]\sum_{k=2}^{n} a_k k\, z^{k-1}\right]$$

Put $\quad \phi(z) = \sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1} + \left[1 + \sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1}\right]\sum_{k=2}^{n} a_k k\, z^{k-1}$. Then we have

$$\mathbf{Re}\left[\frac{z s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] = \frac{1}{|\mathbf{h}_n(z)|^2}\mathbf{Re}[1 + \phi(z)] = \frac{1}{|\mathbf{h}_n(z)|^2}[1 + \mathbf{Re}\,\phi(z)]$$

We have $\quad -|\phi(z)| \le [\mathbf{Re}\,\phi(z)] \le |\phi(z)| \quad$ since $\quad [\mathbf{Re}\,\phi(z)]^2 \le |\phi(z)|^2$.

Thus we have

$$|\mathbf{h}_n(z)|^2\, \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] = 1 + \mathbf{Re}\,\phi(z) \ge 1 - |\phi(z)|$$

Consider

$$\begin{aligned}
|\phi(z)| &= \left|\sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1} + \left[1 + \sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1}\right]\sum_{k=2}^{n} a_k k\, z^{k-1}.\right| \\
&\le \left|\sum_{k=2}^{n} \bar{a}_k\, \bar{z}^{k-1}\right| + \left|\left[1 + \sum_{k=2}^{n} \bar{a}_k\, \bar{z}^{k-1}\right]\sum_{k=2}^{n} a_k k\, z^{k-1}\right| \\
&= \left|\sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1}\right| + \left|1 + \sum_{k=2}^{n} \bar{a}_k k\, \bar{z}^{k-1}\right| \cdot \left|\sum_{k=2}^{n} a_k k\, z^{k-1}\right| \\
&\le \left|\sum_{k=2}^{n} \bar{a}_k \bar{z}^{k-1}\right| + \left[1 + \left|\sum_{k=2}^{n} \bar{a}_k k\, \bar{z}^{k-1}\right|\right] \cdot \left|\sum_{k=2}^{n} a_k k\, z^{k-1}\right| \\
&\le \sum_{k=2}^{n} |\bar{a}_k| \cdot \left|\bar{z}^{k-1}\right| + \left[1 + \sum_{k=2}^{n} |\bar{a}_k| \cdot \left|\bar{z}^{k-1}\right|\right] \cdot \sum_{k=2}^{n} |a_k| \cdot k \cdot \left|z^{k-1}\right| \\
&= \sum_{k=2}^{n} |\bar{a}_k| \cdot |\bar{z}|^{k-1} + \left[1 + \sum_{k=2}^{n} |\bar{a}_k| \cdot |\bar{z}|^{k-1}\right] \cdot \sum_{k=2}^{n} |a_k| \cdot k \cdot |z|^{k-1}
\end{aligned}$$

Put $|z| = r < 1.$ Then

$$|\phi(z)| \le \sum_{k=2}^{n} |\bar{a}_k| \cdot r^{k-1} + \left[1 + \sum_{k=2}^{n} |\bar{a}_k| \cdot r^{k-1}\right] \cdot \sum_{k=2}^{n} |a_k| \cdot k \cdot r^{k-1}$$

$$\le \sum_{k=2}^{n} 1 \cdot r^{k-1} + \left[1 + \sum_{k=2}^{n} 1 \cdot r^{k-1}\right] \cdot \sum_{k=2}^{n} 1 \cdot k\, r^{k-1}$$

$$= r^{-1} \sum_{k=2}^{n} r^k + \left[1 + r^{-1} \sum_{k=2}^{n} r^k\right] \cdot \sum_{k=2}^{n} \frac{d}{dr} r^k$$

$$= \frac{1}{r}\left[\sum_{k=0}^{n} r^k - 1 - r\right] + \left[1 + \frac{1}{r}\left[\sum_{k=0}^{n} r^k - 1 - r\right]\right] \cdot \frac{d}{dr}\left[\sum_{k=0}^{n} r^k - 1 - r\right]$$

$$= \frac{1}{r}\left[\frac{1-r^{n+1}}{1-r} - 1 - r\right] + \left[1 + \frac{1}{r}\left[\frac{1-r^{n+1}}{1-r} - 1 - r\right]\right] \cdot \frac{d}{dr}\left[\frac{1-r^{n+1}}{1-r} - 1 - r\right]$$

$$= \frac{1-r^{n+1}}{r(1-r)} - \frac{1}{r} - 1 + \left[1 + \frac{1-r^{n+1}}{r(1-r)} - \frac{1}{r} - 1\right]\left[\frac{d}{dr}\left(\frac{1-r^{n+1}}{1-r}\right) - 0 - 1\right]$$

$$= \frac{1-r^{n+1}}{r(1-r)} - \frac{1-r}{r(1-r)} - 1 + \left[\frac{1-r^{n+1}}{r(1-r)} - \frac{1-r}{r(1-r)}\right] \cdot \left[\frac{d}{dr}\left(\frac{1-r^{n+1}}{1-r}\right) - 1\right]$$

$$= \frac{1-r^{n+1} - 1 + r}{r(1-r)} - 1 + \left[\frac{1-r^{n+1} - 1 + r}{r(1-r)}\right] \cdot \left[\frac{(1-r)(-(n+1)r^n) - (1-r^{n+1})(-1)}{(1-r)^2} - 1\right]$$

$$= \frac{r-r^{n+1}}{(1-r)^2} - 1 + \left[\frac{r-r^{n+1}}{(1-r)^2}\right]\left[\frac{-(1-r)(n+1)r^n + 1 - r^{n+1}}{(1-r)^2} - 1\right]$$

i.e. $$|\phi(z)| \le \frac{r-r^{n+1}}{(1-r)^2} - 1 + \frac{r-r^{n+1}}{(1-r)^2} \cdot \left[\frac{-(1-r)(n+1)r^n - r^{n+1}}{(1-r)^2} + \frac{1}{(1-r)^2} - 1\right]$$

$$\le \frac{r-r^{n+1}}{(1-r)^2} - 1 - \frac{(r-r^{n+1})\left[(1-r)(n+1)r^n + r^{n+1}\right]}{(1-r)^4} + \frac{r-r^{n+1}}{(1-r)^4} - \frac{r-r^{n+1}}{(1-r)^2}$$

$$\le -1 - \frac{(r-r^{n+1})\left[(1-r)(n+1)r^n + r^{n+1}\right] + r^{n+1}}{(1-r)^4} + \frac{r}{(1-r)^4}$$

Thus

$$1 - |\phi(z)| \ge 1 - \left[-1 - \frac{(r-r^{n+1})\left[(1-r)(n+1)r^n + r^{n+1}\right] + r^{n+1}}{(1-r)^4} + \frac{r}{(1-r)^4}\right]$$

$$= 1 + 1 + \frac{(r-r^{n+1})[(1-r)(n+1)r^n + r^{n+1}] + r^{n+1}}{(1-r)^4} - \frac{r}{(1-r)^4} > 2 + 0 - \frac{1}{(1-r)^4}$$

since $0 < r < 1$ *i.e.* $0 < r^{n+1} < r^n < \cdots < r^2 < r < 1 \quad \Rightarrow \quad 0 < 1 - r, \;\; 0 < r - r^{n+1}, \;\; -r > -1.$

Assume that

$$2 - \frac{1}{(1-r)^4} > 0 \quad \Rightarrow \quad 2 > \frac{1}{(1-r)^4} \quad \Rightarrow \quad (1-r)^4 > \frac{1}{2} \quad \Rightarrow \quad 1 - r > \frac{1}{\sqrt[4]{2}}$$

$$\Rightarrow \quad 1 - \frac{1}{\sqrt[4]{2}} > r \quad \Rightarrow \quad r < 1 - \frac{1}{\sqrt[4]{2}} < 1; \quad \left(1 - \frac{1}{\sqrt[4]{2}} \cong 1 - 0.84 = 0.1591\right)$$

$$\Rightarrow \quad 1 - \frac{1}{\sqrt[4]{2}} > r \quad \Rightarrow \quad r < 1 - \frac{1}{\sqrt[4]{2}} < 1; \quad \left(1 - \frac{1}{\sqrt[4]{2}} \cong 1 - 0.84 = 0.1591\right)$$

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Thus we have

$$|\mathbf{h}_n(z)|^2\ \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] \geq 1-|\phi(z)| > 2-\frac{1}{(1-r)^4} > 0, \qquad \left(|z|=r<1-\frac{1}{\sqrt[4]{2}}<1\right)$$

$$\Rightarrow \qquad \mathbf{Re}\left[\frac{z\,s_n'(z,\mathbf{f})}{s_n(z,\mathbf{f})}\right] > 0, \qquad \left(|z|=r<1-\frac{1}{\sqrt[4]{2}}<1\right).$$

Hence $s_n(z,\mathbf{f})$ are star like functions in $|z|<1-^{-4}\!\!\sqrt{2}.//$

Theorem 8 : Let $\mathbf{L}(z)=z(1-z)^{-1}$. Then $s_n(z,\mathbf{L})$ $(n=2,3,4,\cdots)$ is cap like function in disc $|z|<0.25.$

Proof : $\mathbf{L}(z)=z(1-z)^{-1}=z\sum_{k=0}^{\infty}z^k=\sum_{k=0}^{\infty}z^{k+1}=\sum_{k-1=0}^{\infty}z^{k-1+1}=\sum_{k=1}^{\infty}z^k=z+\sum_{k=2}^{\infty}a_k z^k$

where $a_k=1$ $(k=2,3,4,\cdots)$. Then we have

$$s_n(z,\mathbf{L})=z+\sum_{k=2}^{n}z^k=\sum_{k=1}^{n}z^k=\sum_{k+1=1}^{n}z^{k+1}=z\sum_{k=0}^{n-1}z^{k+1}=z\frac{1-z^n}{1-z}$$

$$\Rightarrow \qquad s_n'(z,\mathbf{L})=\frac{(1-z)[1-(n+1)z^n]-(0-1)[z-z^{n+1}]}{(1-z)^2}$$

$$=\frac{1-z-(n+1)z^n+(n+1)z^{n+1}+z-z^{n+1}}{(1-z)^2}=\frac{1-(n+1)z^n+nz^{n+1}}{(1-z)^2}$$

$$\Rightarrow \qquad \log s_n'(z,\mathbf{L})=\log\left[1-(n+1)z^n+nz^{n+1}\right]-2\log(1-z)$$

By taking the derivative on both sides, we have

$$\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}=\frac{0-(n+1)nz^{n-1}+n(n+1)z^n}{1-(n+1)z^n+nz^{n+1}}-2\frac{-1}{1-z}=\frac{(n+1)nz^{n-1}[-1+z]}{1-(n+1)z^n+nz^{n+1}}+\frac{2}{1-z}$$

$$\Rightarrow \qquad z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}=\frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n+nz^{n+1}}+\frac{2z}{1-z}=\frac{N(z)}{D(z)}+\frac{2z}{1-z}$$

$$\Rightarrow \qquad 1+z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}=1+\frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n+nz^{n+1}}+\frac{2z}{1-z}=\frac{(n+1)nz^n[-1+z]}{1-(n+1)z^n+nz^{n+1}}+\frac{1+z}{1-z}$$

To simplify the notations, put

$$N(z)=(n+1)nz^n[-1+z], \qquad D(z)=1-(n+1)z^n+nz^{n+1}, \qquad \frac{1+z}{1-z}=w=u+\iota v$$

$$\Rightarrow \qquad 1+z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}=\frac{N(z)}{D(z)}+w$$

$$\Rightarrow \qquad \mathbf{Re}\left[1+z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}\right]=\mathbf{Re}\left[\frac{N(z)}{D(z)}+w\right]=\mathbf{Re}\left[\frac{N(z)}{D(z)}\right]+\mathbf{Re}\,w=\mathbf{Re}\left[\frac{N(z)}{D(z)}\right]+u \quad \cdots\cdots\cdots\cdots(1)$$

We have

$$w=\frac{1+z}{1-z} \quad \Leftrightarrow \quad w-wz=1+z \quad \Leftrightarrow \quad w-1=z+wz \quad \Leftrightarrow \quad w-1=(1+w)z$$

Consider

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$$|z| = \frac{1}{4} \quad \Leftrightarrow \quad \left|\frac{w-1}{w+1}\right| = \frac{1}{4} \quad \Leftrightarrow \quad 4|w-1| = |w+1|$$

$$\Leftrightarrow \quad 4|u+\iota v-1| = |u+\iota v+1| \quad \Leftrightarrow \quad 16|u-1+\iota v|^2 = |u+1+\iota v|^2$$

$$\Leftrightarrow \quad 16[(u-1)^2+v^2] = [(u+1)^2+v^2] \quad \Leftrightarrow \quad 16[u^2-2u+1+v^2] = [u^2+2u+1+v^2]$$

$$\Leftrightarrow \quad 16u^2-32u+16+16v^2 = u^2+2u+1+v^2 \quad \Leftrightarrow \quad 15u^2-34u+15+15v^2 = 0$$

$$\Leftrightarrow \quad u^2 - \frac{34}{15}u + 1 + v^2 = 0 \quad \Leftrightarrow \quad u^2 - 2\frac{17}{15}u + \left(\frac{17}{15}\right)^2 - \left(\frac{17}{15}\right)^2 + 1 + v^2 = 0$$

$$\Leftrightarrow \quad \left(u - \frac{17}{15}\right)^2 + v^2 = \frac{289}{225} - 1 = \frac{289-225}{225} = \frac{64}{225} = \left(\frac{8}{15}\right)^2.$$

Consider

$$\mathbf{max}\left(u - \frac{17}{15}\right)^2 = \mathbf{max}\left[\left(\frac{8}{15}\right)^2 - v^2\right] = \left(\frac{8}{15}\right)^2 \qquad i.e. \qquad \mathbf{max} \text{ will exist at } v = 0$$

$$\Rightarrow \quad \left(u - \frac{17}{15}\right)^2 = \left(\frac{8}{15}\right)^2 \quad \Rightarrow \quad u - \frac{17}{15} = \pm\frac{8}{15} \quad \Rightarrow \quad u = \frac{17}{15} \pm \frac{8}{15}$$

$$\Rightarrow \quad u = \frac{17}{15} + \frac{8}{15} = \frac{25}{15} \quad \text{or} \quad u = \frac{17}{15} - \frac{8}{15} = \frac{9}{15}.$$

Hence it is clear that the Mobius **(**Bilinear**)** transformation

$$w = \frac{1+z}{1-z}$$

maps the circle $|z| = 4^{-1}$ in xy-plane into the the circle

$$\left(u - \frac{17}{15}\right)^2 + v^2 = \left(\frac{8}{15}\right)^2$$

in uv-plane such that the line segment AB on u-axis $(v=0)$ is a diameter where

$$A = \left(\frac{9}{15}, 0\right) = \left(\frac{3}{5}, 0\right), \quad \text{and} \quad B = \left(\frac{25}{15}, 0\right) = \left(\frac{5}{3}, 0\right).$$

$$\Rightarrow \quad \frac{9}{15} \le u \le \frac{25}{15} \quad \text{or} \quad \frac{3}{5} \le u \le \frac{5}{3} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2).$$

Observe that $\quad |N(z)| = |(n+1)nz^n[-1+z]| = (n+1)n|z^n|\,|-1+z| \le (n+1)n|z|^n\,[1+|z|]$

Let $\quad |z| < 4^{-1}$. Then $\quad |N(z)| \le (n+1)n|z|^n\,[1+|z|] \le (n+1)n\,(4^{-1})^n\,[1+4^{-1}]$

$i.e.\quad |N(z)| \le (n+1)n\,(4^{-1})^n\,[1+4^{-1}] = (n+1)n\,4^{-n-1}[4+1]$

$i.e.\quad |N(z)| \le 5(n+1)n\,4^{-n-1}.$

Consider $\quad |nz^{n+1} - (n+1)z^n| \le |nz^{n+1}| + |-(n+1)z^n| = n|z|^{n+1} + (n+1)|z|^n$

Let $\quad |z| < 4^{-1}$. Then $\quad n|z|^{n+1} + (n+1)|z|^n \le n4^{-n-1} + (n+1)4^{-n}$

Thus, by transitive law, $\quad |nz^{n+1} - (n+1)z^n| \le n4^{-n-1} + (n+1)4^{-n} < 1$

$$\Rightarrow \quad -|nz^{n+1} - (n+1)z^n| \ge -n4^{-n-1} - (n+1)4^{-n} > -1$$

$$\Rightarrow \quad 1 - |nz^{n+1} - (n+1)z^n| \ge 1 - n4^{-n-1} - (n+1)4^{-n} > 1 - 1 = 0$$

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But $\quad |D(z)| = \left|1+nz^{n+1}-(n+1)z^{n}\right| \geq 1-\left|nz^{n+1}-(n+1)z^{n}\right| \geq 0$

$$\Rightarrow \quad \frac{1}{|D(z)|} \leq \frac{1}{1-n4^{-n-1}-(n+1)4^{-n}} \qquad \text{for} \quad |z|<4^{-1}.$$

Thus we have, for $|z|<4^{-1}$,

$$\left|\frac{N(z)}{D(z)}\right| = \frac{|N(z)|}{|D(z)|} \leq \frac{5(n+1)n\,4^{-n-1}}{1-n4^{-n-1}-(n+1)4^{-n}} = \frac{5(n+1)n}{4^{n+1}-n-(n+1)4} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3).$$

Observe that

$$n=2 \quad \Rightarrow \quad \frac{5(n+1)n}{4^{n+1}-n-(n+1)4} = \frac{5(2+1)2}{4^{2+1}-2-(2+1)4} = \frac{10(3)}{64-2-12} = \frac{10(3)}{50} = \frac{3}{5}.$$

$$\frac{5(n+1)n}{4^{n+1}-n-(n+1)4} \leq \frac{3}{5} \quad \Leftrightarrow \quad \frac{25}{12} \leq \frac{4^{n+1}-n-(n+1)4}{4(n+1)n} = \frac{4^{n}}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \qquad \cdots\cdots\cdots(4).$$

We know that

$$\frac{1}{4(n+1)}<1, \quad \frac{1}{n}<1 \quad \Rightarrow \quad -\frac{1}{4(n+1)}>-1, \quad -\frac{1}{n}>-1 \quad \Rightarrow \quad -\frac{1}{4(n+1)}-\frac{1}{n}>-1-1$$

for any $n=2,3,4,\cdots$

$$n=3 \quad \Rightarrow \quad \frac{4^{n}}{n(n+1)} = \frac{4^{3}}{3(3+1)} = \frac{64}{12} > \frac{25}{12}.$$

Observe that for all integers k

$$\begin{cases} \dfrac{4^{k+1}}{(k+1)(k+2)} > \dfrac{4^{k}}{k(k+1)} \quad \Leftrightarrow \quad \dfrac{4^{k}4}{(k+1)(k+2)} > \dfrac{4^{k}}{k(k+1)} \quad \Leftrightarrow \quad \dfrac{4}{k+2} > \dfrac{1}{k} \\ \Leftrightarrow \quad 4k>k+2 \quad \Leftrightarrow \quad 3k>2 \end{cases}$$

Since $3k>2$ for all integers $k \geq 1$, we have

$$\frac{4^{k+1}}{(k+1)(k+2)} > \frac{4^{k}}{k(k+1)} > \cdots > \frac{4^{3}}{3(3+1)} = \frac{64}{12}.$$

Thus we have

$$\frac{4^{n}}{n(n+1)} \geq \frac{64}{12} \quad (n=3,4,5,\cdots) \quad \Rightarrow \quad \frac{4^{n}}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \geq \frac{64}{12} - 2 = \frac{40}{12} > \frac{25}{12}$$

Thus, from (4), we have

$$\frac{5(n+1)n}{4^{n+1}-n-(n+1)4} \leq \frac{3}{5}.$$

Thus, from (3), we have

$$\left|\frac{N(z)}{D(z)}\right| \leq \frac{5(n+1)n}{4^{n+1}-n-(n+1)4} \leq \frac{3}{5} \quad \Rightarrow \quad -\frac{3}{5} \leq -\left|\frac{N(z)}{D(z)}\right|.$$

We know that $|\mathbf{f}(z)|^{2} = [\mathbf{Re}\,\mathbf{f}(z)]^{2} + [\mathbf{Im}\,\mathbf{f}(z)]^{2} \geq [\mathbf{Re}\,\mathbf{f}(z)]^{2}$

$$\Rightarrow \quad [\mathbf{Re}\,\mathbf{f}(z)]^{2} \leq |\mathbf{f}(z)|^{2} \quad \Rightarrow \quad -|\mathbf{f}(z)| \leq \mathbf{Re}\,\mathbf{f}(z) \leq |\mathbf{f}(z)|.$$

Hence we have

$$-\frac{3}{5}\le -\left|\frac{N(z)}{D(z)}\right| < \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] \quad \Rightarrow \quad \mathbf{Re}\left[\frac{N(z)}{D(z)}\right] > -\frac{3}{5}.$$

Therefore, from (1) and (2), we have

$$\mathbf{Re}\left[1+z\frac{s_n''(z,\mathbf{L})}{s_n'(z,\mathbf{L})}\right]=\mathbf{Re}\left[\frac{N(z)}{D(z)}\right]+u>-\frac{3}{5}+\frac{3}{5}=0.$$

It remains to show that 0.25 is maximal radius. This is seen for $s_2(z,\mathbf{L})=z+z^2$. Then

$$1+z\frac{s_2''(z,\mathbf{L})}{s_2'(z,\mathbf{L})}=1+z\frac{0+2}{1+2z}=\frac{1+4z}{1+2z}$$

has singularity at $z=-0.25$, and thus analytic within $|z|<0.25$.

Clearly $s_n(z,\mathbf{L})=z+\sum_{k=2}^{n} z^k$ is analytic within $|z|<0.25$, and $s_n(0,\mathbf{L})=0$,

Since $s_n'(z,\mathbf{L})=1+\sum_{k=2}^{n} k\,z^{k-1}$, we have $s_n'(0,\mathbf{L})=1+\sum_{k=2}^{n} k\,0^{k-1}=1$.

Hence $s_n(z,\mathbf{L})$ is cap like function in the open disc $|z|<0.25.//$

Definition 7 : *Hadamard product* (*Convolution*) of two analytic functions $\mathbf{f}(z)=\sum_{k=0}^{\infty} a_k z^k$ in the open disc $|z|<r_1$ and $\mathbf{g}(z)=\sum_{k=0}^{\infty} b_k z^k$ in the open disc $|z|<r_2$ is denoted by $\mathbf{f}*\mathbf{g}$ and is defined as an anlytic function $(\mathbf{f}*\mathbf{g})(z)=\sum_{k=0}^{\infty} a_k b_k z^k$ in the open disc $|z|<r_1 r_2$.

Theorem 9 : Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty} a_k z^k$, and $\mathbf{g}(z)=z+\sum_{k=2}^{\infty} b_k z^k$ are univalent functions. Then $(\mathbf{f}*\mathbf{g})(z)$ is cap like function in the open disc $|z|<6^{-1}<1$, and is univalent in the open disc $\{z/|z|<3^{-1}<1\}$.

Proof : By **Theorem 1,** $|a_k|\le 1$ $(k=2,3,4,\cdots)$, and $|b_k|\le 1$ $(k=2,3,4,\cdots)$ since $\mathbf{f}(z)$, $\mathbf{g}(z)$ are univalent in the unit open disc $|z|<1$.

Now $(\mathbf{f}*\mathbf{g})(z)=z+\sum_{k=2}^{\infty} a_k b_k z^k$

Let us consider

$$\mathbf{Re}\left[1+\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]=1+\mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right].$$

We know that

$$\left[\mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]\right]^2\le\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|^2 \quad\Rightarrow\quad -\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|\le\mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]\le\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|$$

$$\Rightarrow\quad 1-\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|\le 1+\mathbf{Re}\left[\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]=\mathbf{Re}\left[1+\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]$$

$$\Rightarrow\quad \mathbf{Re}\left[1+\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right]\ge 1-\left|\frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right|=1-\frac{|z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|}=\frac{|(\mathbf{f}*\mathbf{g})'(z)|-|z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|}$$

We have $(\mathbf{f}*\mathbf{g})'(z)=1+\sum_{k=2}^{\infty} a_k b_k k z^{k-1}$

$$\Rightarrow\quad (\mathbf{f}*\mathbf{g})''(z)=\sum_{k=2}^{\infty} a_k b_k k(k-1)z^{k-2} \quad\Rightarrow\quad z(\mathbf{f}*\mathbf{g})''(z)=\sum_{k=2}^{\infty} a_k b_k k(k-1)z^{k-1}$$

$$\Rightarrow\quad |z(\mathbf{f}*\mathbf{g})''(z)|=\left|\sum_{k=2}^{\infty} a_k b_k k(k-1)z^{k-1}\right|\le\sum_{k=2}^{\infty}|a_k||b_k|k(k-1)|z|^{k-1}\le\sum_{k=2}^{\infty}1k(k-1)|z|^{k-1}$$

Put $|z| = r \quad \Rightarrow \quad |z(\mathbf{f}*\mathbf{g})''(z)| \le \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1}.$

By Triangle inequality, $\quad 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \le |1 + \sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| = |(\mathbf{f}*\mathbf{g})'(z)|$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| \ge 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}|$

But $\quad |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \le \sum_{k=2}^{\infty} |a_k||b_k| k |z|^{k-1} \le \sum_{k=2}^{\infty} 1k|z|^{k-1} = \sum_{k=2}^{\infty} k\, r^{k-1}$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| \ge 1 - |\sum_{k=2}^{\infty} a_k b_k k\, z^{k-1}| \ge 1 - \sum_{k=2}^{\infty} k\, r^{k-1}$

$\Rightarrow \quad |(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 1 - \sum_{k=2}^{\infty} k\, r^{k-1} - \sum_{k=2}^{\infty}(k^2-k)\, r^{k-1} = 1 - \sum_{k=2}^{\infty} k^2\, r^{k-1}.$

We have

$$\sum_{k=2}^{\infty} k^2 r^{k-1} = \sum_{k=2}^{\infty}[k(k-1)+k] r^{k-1} = \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1}$$
$$= \sum_{k=2}^{\infty} k(k-1) r^{k-1} + \sum_{k=2}^{\infty} k\, r^{k-1} = r\sum_{k=2}^{\infty} k(k-1) r^{k-2} + \sum_{k=2}^{\infty} k\, r^{k-1}.$$

i.e. $\quad \sum_{k=2}^{\infty} k^2 r^{k-1} = r\sum_{k=2}^{\infty}\frac{d^2}{dr^2} r^k + \sum_{k=2}^{\infty}\frac{d}{dr} r^k = r\frac{d^2}{dr^2}\sum_{k=2}^{\infty} r^k + \frac{d}{dr}\sum_{k=2}^{\infty} r^k = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\sum_{k=2}^{\infty} r^k$

i.e. $\quad \sum_{k=2}^{\infty} k^2 r^{k-1} = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\left[\sum_{k=0}^{\infty} r^k - 1 - r\right] = \left[r\frac{d^2}{dr^2} + \frac{d}{dr}\right]\left[\frac{1}{1-r} - 1 - r\right]$

i.e. $\quad \sum_{k=2}^{\infty} k^2 r^{k-1} = r\frac{2}{(1-r)^3} + \frac{1}{(1-r)^2} - 1 = \frac{2r+1-r}{(1-r)^3} - 1 = \frac{r+1}{(1-r)^3} - 1$

Thus we have

$$|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 1 - \sum_{k=2}^{\infty} k^2 r^{k-1} = 1 - \frac{r+1}{(1-r)^3} + 1 = 2 - \frac{r+1}{(1-r)^3}$$

We have

$$2 - \frac{r+1}{(1-r)^3} > 0 \quad \Leftrightarrow \quad 2 > \frac{r+1}{(1-r)^3} \quad \Leftrightarrow \quad 2(1-r)^3 > r+1$$

$\Leftrightarrow \quad 2[1 - r^3 - 3r(1-r)] > r+1 \quad \Leftrightarrow \quad 2 - 2r^3 - 6r + 6r^2 > r+1$

$\Leftrightarrow \quad 1 - 7r + 6r^2 - 2r^3 > 0 \quad \Leftrightarrow \quad 1 - 7\frac{1}{R} + 6\frac{1}{R^2} - 2\frac{1}{R^3} > 0 \quad (Rr = 1,\ R > 1 \text{ since } r < 1)$

$\Leftrightarrow \quad R^3 - 7R^2 + 6R - 2 > 0 \quad \Leftrightarrow \quad R^3 - 7R^2 + 6R > 2 \quad \Leftrightarrow \quad R(R^2 - 7R + 6) > 2$

$\Leftrightarrow \quad R(R-1)(R-6) > 2 \quad \Leftrightarrow \quad R > 6 \quad (\because R > 1)$

$\Leftrightarrow \quad \frac{1}{r} > 6 \quad \Rightarrow \quad r < \frac{1}{6} \quad \Rightarrow \quad |z| < \frac{1}{6} < 1.$

Thus, for $|z| < 6^{-1} < 1$, we have

$$|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)| \ge 2 - \frac{r+1}{(1-r)^3} > 0$$

$$\Rightarrow \quad \mathbf{Re}\left[1 + \frac{z(\mathbf{f}*\mathbf{g})''(z)}{(\mathbf{f}*\mathbf{g})'(z)}\right] \ge \frac{|(\mathbf{f}*\mathbf{g})'(z)| - |z(\mathbf{f}*\mathbf{g})''(z)|}{|(\mathbf{f}*\mathbf{g})'(z)|} > 0.$$

Hence $(\mathbf{f}*\mathbf{g})(z)$ is cap like function in the open disc $|z| < 6^{-1} < 1$.

Let $z_1 \ne z_2$, we have $z_1 - z_2 \ne 0 \quad \Rightarrow \quad |z_1 - z_2| \ne 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|.$

Let $\rho = |z_1| \le |z_2| = r < 1 \quad \Rightarrow \quad 0 \le r - \rho = |z_2| - |z_1| < |z_1 - z_2|$ by triangle inequality.

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Consider

$$\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)=\left[z_1+\sum_{k=2}^{\infty}a_kb_kkz_1^k\right]-\left[z_2+\sum_{k=2}^{\infty}ka_kb_kz_2^k\right]$$
$$=z_1-z_2+\left[\sum_{k=2}^{\infty}a_kb_kz_1^k-\sum_{k=2}^{\infty}a_kb_kz_2^k\right]=z_1-z_2+\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]$$

$$\Rightarrow \quad |\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|=\left|z_1-z_2+\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|$$

By triangle inequality, $\left|z_1-z_2+\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|\ge|z_1-z_2|-\left|\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|$

$$\Rightarrow \quad |\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|\ge|z_1-z_2|-\left|\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|>r-\rho-\left|\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|$$

Again by triangle inequality; and since $|a_k|\le 1$, $|b_k|\le 1$; we have

$$\left|\sum_{k=2}^{n}a_kb_k\left[z_1^k-z_2^k\right]\right|\le\sum_{k=2}^{n}\left|a_kb_k\left[z_1^k-z_2^k\right]\right|=\sum_{k=2}^{n}|a_k||b_k||z_1^k-z_2^k|$$
$$\le\sum_{k=2}^{n}1(1)\left[|z_1^k|+|z_2^k|\right]$$
$$=\sum_{k=2}^{n}\left[|z_1|^k+|z_2|^k\right]\le\sum_{k=2}^{n}\left[r^k+r^k\right].$$

i.e. $\left|\sum_{k=2}^{\infty}ka_k\left[z_1^k-z_2^k\right]\right|\le 2\sum_{k=2}^{\infty}r^k=2\dfrac{r^2}{1-r} \quad\Rightarrow\quad -\left|\sum_{k=2}^{\infty}ka_k\left[z_1^k-z_2^k\right]\right|\ge-\dfrac{2r^2}{1-r}$

$$\Rightarrow \quad |\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|>r-\rho-\left|\sum_{k=2}^{\infty}a_kb_k\left[z_1^k-z_2^k\right]\right|\ge r-\rho-\frac{2r^2}{1-r}=r\left[1-\frac{2r}{1-r}\right]-\rho$$

By taking limit $\rho=|z_1|\to 0$ on both sides since either $0=|z_1|<|z_2|$ or $0<|z_1|\le|z_2|$,

$$|\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|>r\left[1-\frac{2r}{1-r}\right]-\lim_{\rho\to 0}\rho=r\left[1-\frac{2r}{1-r}\right]-0=r\left[1-\frac{2r}{1-r}\right].$$

Hence, for $|\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|>0$, we must have

$$1-\frac{2r}{1-r}>0 \quad\Leftrightarrow\quad 1>\frac{2r}{1-r} \quad\Leftrightarrow\quad 1-r>2r \quad (\because 0<r<1 \ \ i.e.\ \ 0<1-r)$$

$$\Leftrightarrow \quad 1>5r \quad\Leftrightarrow\quad 3r<1 \quad\Leftrightarrow\quad r<3^{-1}$$

Thus, for $0\le\rho=|z_1|\le|z_2|=r<3^{-1}$, we have

$$|\mathbf{(f*g)}(z_1)-\mathbf{(f*g)}(z_2)|>r\left[1-\frac{2r}{1-r}\right]-\rho>0 \quad i.e. \quad \mathbf{(f*g)}(z_1)\ne\mathbf{(f*g)}(z_2).$$

Thus $\mathbf{(f*g)}(z)$ is univalent in the open disc $\{z/|z|<3^{-1}<1\}$.//

Theorem 10 : Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_kz^k$ is univalent function. Then $\mathbf{f}(z)$ is star like function in the open disc $\{z/|z|<c<1\}$ where $c=1-3^{-1}\sqrt{6}$.

Proof : Let $\mathbf{g}(z)=z+\sum_{k=2}^{\infty}k^{-1}z^k=\sum_{k=1}^{\infty}k^{-1}z^k$. Then clearly $\mathbf{g}(0)=0$, and $\mathbf{g}'(0)=1$.

And $\mathbf{g}'(z)=\sum_{k=1}^{\infty}z^{k-1}=\sum_{k+1=1}^{\infty}z^{k+1-1}=\sum_{k=0}^{\infty}z^k=(1-z)^{-1} \quad\Rightarrow\quad \mathbf{g}''(z)=(1-z)^{-2}$

$$\Rightarrow \quad 1+\frac{z\mathbf{g}''(z)}{\mathbf{g}'(z)}=1+\frac{z(1-z)^{-2}}{(1-z)^{-1}}=1+\frac{z}{1-z}=\frac{1-z+z}{1-z}=\frac{1}{1-z}=\frac{1}{1-(x+\iota y)}, \quad (\because z=x+\iota y).$$

i.e. $1+\dfrac{z\mathbf{g}''(z)}{\mathbf{g}'(z)}=\dfrac{1}{(1-x)-\iota y}=\dfrac{(1-x)+\iota y}{(1-x)^2+y^2} \quad\Rightarrow\quad \mathbf{Re}\left[1+\dfrac{z\mathbf{g}''(z)}{\mathbf{g}'(z)}\right]=\dfrac{(1-x)}{(1-x)^2+y^2}.$

For $|z|<1$, we have $x^2\le x^2+y^2=|z|^2<1 \quad\Rightarrow\quad -1<x<1 \quad\Rightarrow\quad 0<1-x$

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Thus we have

$$\mathbf{Re}\left[1+\frac{z\,\mathbf{g}''(z)}{\mathbf{g}'(z)}\right]=\frac{(1-x)}{(1-x)^2+y^2}>0.$$

$\Rightarrow$ $\mathbf{g}(z)$ is cap like function.

Let $z_1 \neq z_2 \quad \Rightarrow \quad z_1 - z_2 \neq 0 \quad \Rightarrow \quad |z_1 - z_2| \neq 0 \quad \Rightarrow \quad 0 < |z_1 - z_2|$.

Let $\rho = |z_1| \leq |z_2| = r < 1 \quad \Rightarrow \quad r - \rho = |z_2| - |z_1| < |z_1 - z_2|$ by triangle inequality.

Consider

$$\mathbf{g}(z_1)-\mathbf{g}(z_2) = \left[z_1+\textstyle\sum_{k=2}^{\infty} a_k k^{-1} z_1^k\right]-\left[z_2+\textstyle\sum_{k=2}^{\infty} a_k k^{-1} z_2^k\right]$$

$$= z_1 - z_2 + \left[\textstyle\sum_{k=2}^{\infty} a_k k^{-1} z_1^k - \textstyle\sum_{k=2}^{\infty} a_k k^{-1} z_2^k\right] = z_1 - z_2 + \textstyle\sum_{k=2}^{\infty} a_k k^{-1}\left[z_1^k - z_2^k\right]$$

By triangle inequality, $\left|z_1 - z_2 + \sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| \geq |z_1 - z_2| - \left|\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right|$

$$\Rightarrow \quad |\mathbf{g}(z_1)-\mathbf{g}(z_2)| \geq |z_1 - z_2| - \left|\textstyle\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| > r - \rho - \left|\textstyle\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right|$$

By BEIRBERBACH conjecture $|a_k| \leq k$ since $\mathbf{f}(z) \in \mathbf{U}$.

Again by triangle inequality, we have

$$\left|\textstyle\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| \leq \textstyle\sum_{k=2}^{\infty} \left|a_k k^{-1}[z_1^k - z_2^k]\right| = \textstyle\sum_{k=2}^{\infty} |a_k| \cdot k^{-1} \cdot |z_1^k - z_2^k|$$

$$\leq \textstyle\sum_{k=2}^{\infty} k\, k^{-1}\left[|z_1^k| + |z_2^k|\right]$$

i.e. $\left|\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| \leq \sum_{k=2}^{\infty}\left[|z_1|^k + |z_2|^k\right] \leq \sum_{k=2}^{\infty}\left[r^k + r^k\right]$ since $|z_1| \leq |z_2| = r$.

i.e. $\left|\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| \leq \sum_{k=2}^{\infty} 2r^k = 2\sum_{k=2}^{\infty} r^k = 2\dfrac{r^2}{1-r}$

Thus we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > r - \rho - \left|\textstyle\sum_{k=2}^{\infty} a_k k^{-1}[z_1^k - z_2^k]\right| \geq r - \rho - \frac{2r^2}{1-r} = r\left[1-\frac{2r}{1-r}\right]-\rho.$$

By taking limit $\rho = |z_1| \to 0$ on both sides since either $0 = |z_1| < |z_2|$ or $0 < |z_1| \leq |z_2|$,

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > r\left[1-\frac{2r}{1-r}\right] - \mathbf{lim}_{\rho\to 0}\,\rho = r\left[1-\frac{2r}{1-r}\right] - 0 = r\left[1-\frac{2r}{1-r}\right].$$

Hence, for $|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > 0$, we must have

$$1-\frac{2r}{1-r} > 0 \quad \Leftrightarrow \quad 1 > \frac{2r}{1-r} \quad \Leftrightarrow \quad 1 - r > 2r \quad (\because 0 < r < 1 \quad i.e. \quad 0 < 1 - r)$$

$$\Leftrightarrow \quad 1 > 5r \quad \Leftrightarrow \quad 3r < 1 \quad \Leftrightarrow \quad r < 3^{-1}.$$

Thus, for $0 \leq \rho = |z_1| \leq |z_2| = r < 3^{-1}$, we have

$$|\mathbf{g}(z_1)-\mathbf{g}(z_2)| > r\left[1-\frac{2r}{1-r}\right] - \rho > 0 \quad i.e. \quad |\mathbf{g}(z_1)-\mathbf{g}(z_2)| \neq 0 \quad i.e. \quad \mathbf{g}(z_1)-\mathbf{g}(z_2) \neq 0.$$

Hence $\mathbf{g}(z)$ is univalent function in the disc $|z| < 3^{-1} < 1$.

Convolution of $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} k^{-1} z^k$ is $(\mathbf{f} * \mathbf{g})(z) = z + \sum_{k=2}^{\infty} a_k k^{-1} z^k$.

By **Theorem 9**, we say that $(\mathbf{f} * \mathbf{g})(z)$ is cap like function in the open disc $|z| < 6^{-1} < 1$, and $(\mathbf{f} * \mathbf{g})(z)$ is univalent in the open disc $\{z/\,|z| < 3^{-1} < 1\}$.

$(\mathbf{f} * \mathbf{g})'(z) = 1 + \sum_{k=2}^{\infty} a_k k^{-1} k z^{k-1} \quad \Rightarrow \quad z(\mathbf{f} * \mathbf{g})'(z) = z + \sum_{k=2}^{\infty} a_k z^k = \mathbf{f}(z).$

By **Theorem 4,** $\mathbf{f}(z) = z(\mathbf{f} * \mathbf{g})'(z)$ is star like function in $\{z/ |z| < 1 - 3^{-1}\sqrt{6} = 0.1835 < 1\}$.//

Definition 8 : An analytic function $\mathbf{f}(z)$ in $\{z/|z|<1\}$ with $\mathbf{f}(0)=0,\ \mathbf{f}'(0)=1$ is said to be close-to-cap like function if there is a univalnt cap like function $\mathbf{g}(z)$ in $\{z/|z|<1\}$ such that

$$\mathbf{Re}\left[\frac{\mathbf{f}'(z)}{\mathbf{g}'(z)}\right] > 0, \quad |z|<1.$$

Theorem 11 : Let $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is close-to-cap like function. Then $\mathbf{g}(z)$ is univalnt.

Proof : Let $\mathbf{g}(z)$ is close-to-cap like function. Then there is a univalnt cap like function $\mathbf{h}(z)$ in $\{z/|z|<1\}$ such that

$$\mathbf{Re}\left[\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}\right] > 0, \quad |z|<1.$$

Let $|z_0|<1$

$$\Rightarrow \quad \mathbf{Re}\left[\frac{\mathbf{g}'(z_0)}{\mathbf{h}'(z_0)}\right] > 0$$

$$\Rightarrow \quad \mathbf{Re}\left[\frac{\lim\limits_{z\to z_0}\dfrac{\mathbf{g}(z)-\mathbf{g}(z_0)}{z-z_0}}{\lim\limits_{z\to z_0}\dfrac{\mathbf{h}(z)-\mathbf{h}(z_0)}{z-z_0}}\right] > 0 \quad \Rightarrow \quad \mathbf{Re}\left[\lim_{z\to z_0}\frac{\dfrac{\mathbf{g}(z)-\mathbf{g}(z_0)}{z-z_0}}{\dfrac{\mathbf{h}(z)-\mathbf{h}(z_0)}{z-z_0}}\right] > 0$$

$$\Rightarrow \quad \mathbf{Re}\left[\lim_{z\to z_0}\frac{\mathbf{g}(z)-\mathbf{g}(z_0)}{\mathbf{h}(z)-\mathbf{h}(z_0)}\right] > 0, \quad z \neq z_0 \quad \Rightarrow \quad \lim_{z\to z_0}\mathbf{Re}\left[\frac{\mathbf{g}(z)-\mathbf{g}(z_0)}{\mathbf{h}(z)-\mathbf{h}(z_0)}\right] > 0, \quad z \neq z_0$$

$$\Rightarrow \quad \mathbf{Re}\left[\frac{\mathbf{g}(z)-\mathbf{g}(z_0)}{\mathbf{h}(z)-\mathbf{h}(z_0)}\right] > 0, \quad z \neq z_0.$$

But $\mathbf{h}(z) \neq \mathbf{h}(z_0)$ for $z \neq z_0$ since $\mathbf{h}(z)$ is a univalnt function.

$\Rightarrow \quad \mathbf{g}(z)-\mathbf{g}(z_0) \neq 0$ for $z \neq z_0 \quad \Rightarrow \quad \mathbf{g}(z) \neq \mathbf{g}(z_0)$ for $z \neq z_0$.

Thus $\mathbf{g}(z)$ is a univalnt function.//

Theorem 12 : Let $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is close-to-cap like function, and $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then $(\mathbf{f} * \mathbf{g})(z)$ is close-to-cap like function provided $|(\mathbf{f} * \mathbf{g})'(z) - \mathbf{g}'(z) + 1| \geq 1$.

Proof : Let $\mathbf{g}(z)$ is close-to-cap like function. Then $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$; and there is a cap like function $\mathbf{h}(z) = z + \sum_{k=2}^{\infty} c_k z^k$. in $\{z/|z|<1\}$ such that

$$\mathbf{Re}\left[\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}\right] > 0 \quad |z|<1.$$

Now $\mathbf{h}'(z) = 1 + \sum_{k=2}^{\infty} c_k k z^{k-1}$.

Convolution of $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$, and $\mathbf{g}(z)=z+\sum_{k=2}^{\infty}b_k z^k$ is $(\mathbf{f}*\mathbf{g})(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k$.

Consider $(\mathbf{f}*\mathbf{g})'(z)-\mathbf{g}'(z)=\left[1+\sum_{k=2}^{\infty}a_k b_k kz^{k-1}\right]-\left[1+\sum_{k=2}^{\infty}b_k kz^{k-1}\right]=\sum_{k=2}^{\infty}(a_k b_k-b_k)kz^{k-1}$

Without loss of generality, Put $c_k=a_k b_k-b_k$ for $k=2,3,4,\cdots$

$$\Rightarrow \quad (\mathbf{f}*\mathbf{g})'(z)-\mathbf{g}'(z)=\sum_{k=2}^{\infty}(a_k b_k-b_k)kz^{k-1}=\sum_{k=2}^{\infty}c_k kz^{k-1}=\mathbf{h}'(z)-1$$

$$\Rightarrow \quad \frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}-\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}=\frac{\mathbf{h}'(z)-1}{\mathbf{h}'(z)}=1-\frac{1}{\mathbf{h}'(z)} \qquad i.e \qquad \frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}=\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}+1-\frac{1}{\mathbf{h}'(z)}$$

$$\Rightarrow \quad \mathbf{Re}\left[\frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}\right]=\mathbf{Re}\left[\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}+1-\frac{1}{\mathbf{h}'(z)}\right]=\mathbf{Re}\left[\frac{\mathbf{g}'(z)}{\mathbf{h}'(z)}\right]+1-\mathbf{Re}\left[\frac{1}{\mathbf{h}'(z)}\right]>0+1-\mathbf{Re}\left[\frac{1}{\mathbf{h}'(z)}\right]$$

$$i.e. \quad \mathbf{Re}\left[\frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}\right]>1-\mathbf{Re}\left[\frac{1}{\mathbf{h}'(z)}\right]=1-\mathbf{Re}\left[\frac{\overline{\mathbf{h}'(z)}}{\mathbf{h}'(z)\times\overline{\mathbf{h}'(z)}}\right]=1-\mathbf{Re}\left[\frac{\overline{\mathbf{h}'(z)}}{|\mathbf{h}'(z)|^2}\right]=1-\frac{\mathbf{Re}\,\overline{\mathbf{h}'(z)}}{|\mathbf{h}'(z)|^2}$$

Let $\mathbf{h}'(z)=u(x,y)+\iota v(x,y) \quad \Rightarrow \quad \overline{\mathbf{h}'(z)}=u(x,y)-\iota v(x,y)$

$$\Rightarrow \quad |\mathbf{h}'(z)|^2=u^2+v^2, \quad \text{and} \quad \mathbf{Re}\,\overline{\mathbf{h}'(z)}=u.$$

Thus we have

$$\mathbf{Re}\left[\frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}\right]>1-\frac{1}{|\mathbf{h}'(z)|^2}\mathbf{Re}\,\overline{\mathbf{h}'(z)}=1-\frac{u}{u^2+v^2}=\frac{u^2+v^2-u}{u^2+v^2}.$$

$u<0 \quad \Rightarrow \quad u^2+v^2-u>0.$

$0\le u\le 1\le u^2+v^2$ since $u^2+v^2=|\mathbf{h}'(z)|^2=|(\mathbf{f}*\mathbf{g})'(z)-\mathbf{g}'(z)+1|\ge 1$

$\Rightarrow \quad 0\le u^2+v^2-u \quad i.e. \quad u^2+v^2-u>0$

$1<u \quad \Rightarrow \quad u<u^2\le u^2+v^2 \quad \Rightarrow \quad 0<u^2+v^2-u \quad i.e. \quad u^2+v^2-u>0.$

Thus we have

$$\mathbf{Re}\left[\frac{(\mathbf{f}*\mathbf{g})'(z)}{\mathbf{h}'(z)}\right]>\frac{u^2+v^2-u}{u^2+v^2}\ge 0.//$$

Theorem 13 : If $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ is close-to-cap like function in $\{z/|z|<1\}$, then $s_n(z,\mathbf{f})$ are close-to-cap like functions in $|z|<1-\sqrt[-4]{2}$.

Proof : Let $\mathbf{f}(z)=z+\sum_{k=2}^{\infty}a_k z^k$ is close-to-cap like function in $\{z/|z|<1\}$. Then there is a cap like and univalent function $\mathbf{h}(z)$ such that

$$\mathbf{Re}\left[\frac{\mathbf{f}'(z)}{\mathbf{h}'(z)}\right]>0, \quad |z|<1.$$

Clearly $|a_k|\le 1$, $|\bar{c}_k|=|c_k|\le 1$ since $\mathbf{f}(z),\mathbf{h}(z)$ are univalent.

Partial sum of $\mathbf{f}(z)$ is $s_n(z,\mathbf{f})=z+\sum_{k=2}^{n}a_k z^k$.

Consider

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$$\mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right]=\mathbf{Re}\left[\frac{\overline{\mathbf{h}'(z)}\,s_n'(z,\mathbf{f})}{\mathbf{h}'(z)\,\overline{\mathbf{h}'(z)}}\right]=\mathbf{Re}\left[\frac{\overline{\mathbf{h}'(z)}\left[1+\sum_{k=2}^{n}a_k k\,z^{k-1}\right]}{|\mathbf{h}'(z)|^2}\right]$$

i.e. $$\mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right]=\frac{1}{|\mathbf{h}'(z)|^2}\mathbf{Re}\left[\,\overline{\mathbf{h}'(z)}+\overline{\mathbf{h}'(z)}\sum_{k=2}^{n}a_k k\,z^{k-1}\right]$$

i.e. $$\mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right]=\frac{1}{|\mathbf{h}'(z)|^2}\mathbf{Re}\left[1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}+\left[1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right]\sum_{k=2}^{n}a_k k\,z^{k-1}\right]$$

Put $\phi(z)=\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}+\left[1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right]\sum_{k=2}^{n}a_k k\,z^{k-1}$.

$$\Rightarrow\quad \mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right]==\frac{1}{|\mathbf{h}'(z)|^2}\mathbf{Re}[1+\phi(z)]=\frac{1}{|\mathbf{h}'(z)|^2}[1+\mathbf{Re}\,\phi(z)]$$

We have $-|\phi(z)|\le[\mathbf{Re}\,\phi(z)]\le|\phi(z)|$ since $[\mathbf{Re}\,\phi(z)]^2\le|\phi(z)|^2$.

Thus we have

$$|\mathbf{h}'(z)|^2\,\mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right]=1+\mathbf{Re}\,\phi(z)\ge 1-|\phi(z)|$$

Consider

$$\begin{aligned}
|\phi(z)| &= \left|\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}+\left[1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right]\sum_{k=2}^{n}a_k k\,z^{k-1}\right| \\
&\le \left|\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right|+\left|\left[1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right]\sum_{k=2}^{n}a_k k\,z^{k-1}\right| \\
&= \left|\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right|+\left|1+\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right|\cdot\left|\sum_{k=2}^{n}a_k k\,z^{k-1}\right| \\
&\le \left|\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right|+\left[1+\left|\sum_{k=2}^{\infty}\overline{c}_k k\,\overline{z}^{k-1}\right|\right]\cdot\left|\sum_{k=2}^{n}a_k k\,z^{k-1}\right| \\
&\le \sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot|\overline{z}^{k-1}|+\left[1+\sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot|\overline{z}^{k-1}|\right]\cdot\sum_{k=2}^{n}|a_k|\cdot k\cdot|z^{k-1}| \\
&= \sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot|\overline{z}|^{k-1}+\left[1+\sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot|\overline{z}|^{k-1}\right]\cdot\sum_{k=2}^{n}|a_k|\cdot k\cdot|z|^{k-1} \\
&= \sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot r^{k-1}+\left[1+\sum_{k=2}^{\infty}|\overline{c}_k|\cdot k\cdot r^{k-1}\right]\cdot\sum_{k=2}^{n}|a_k|\cdot k\cdot r^{k-1} \\
&\le \sum_{k=2}^{\infty}1\cdot k\cdot r^{k-1}+\left[1+\sum_{k=2}^{\infty}1\cdot k\cdot r^{k-1}\right]\cdot\sum_{k=2}^{n}1\cdot k\,r^{k-1} \\
&= \sum_{k=2}^{\infty}\frac{d}{dr}r^k+\left[1+\sum_{k=2}^{\infty}\frac{d}{dr}r^k\right]\cdot\sum_{k=2}^{n}\frac{d}{dr}r^k=\frac{d}{dr}\sum_{k=2}^{\infty}r^k+\left[1+\frac{d}{dr}\sum_{k=2}^{\infty}r^k\right]\cdot\frac{d}{dr}\sum_{k=2}^{n}r^k \\
&= \frac{d}{dr}\frac{r^2}{1-r}+\left[1+\frac{d}{dr}\frac{r^2}{1-r}\right]\cdot\frac{d}{dr}\left[\sum_{k=0}^{n}r^k-1-r\right] \\
&= \frac{(1-r)2r-r^2(-)}{(1-r)^2}+\left[1+\frac{(1-r)2r-r^2(-)}{(1-r)^2}\right]\cdot\frac{d}{dr}\left[\frac{1-r^{n+1}}{1-r}-1-r\right] \\
&= \frac{2r-r^2}{(1-r)^2}+\left[1+\frac{2r-r^2}{(1-r)^2}\right]\cdot\left[\frac{(1-r)(-(n+1)r^n)-(1-r^{n+1})(-1)}{(1-r)^2}-0-1\right] \\
&= \frac{2r-r^2}{(1-r)^2}+\frac{(1+r^2-2r)+(2r-r^2)}{(1-r)^2}\left[\frac{-(1-r)(n+1)r^n+1-r^{n+1}}{(1-r)^2}-1\right] \\
&= \frac{2r-r^2}{(1-r)^2}+\frac{1}{(1-r)^2}\cdot\left[\frac{-(1-r)(n+1)r^n-r^{n+1}}{(1-r)^2}+\frac{1}{(1-r)^2}-1\right]
\end{aligned}$$

$$i.e. \quad |\phi(z)| \le \frac{2r-r^2}{(1-r)^2} - \frac{(1-r)(n+1)r^n + r^{n+1}}{(1-r)^4} + \frac{1}{(1-r)^4} - \frac{1}{(1-r)^2}, \qquad |z| = r.$$

$$\Rightarrow \qquad 1-|\phi(z)| \ge 1 - \left[\frac{2r-r^2}{(1-r)^2} - \frac{(1-r)(n+1)r^n + r^{n+1}}{(1-r)^4} + \frac{1}{(1-r)^4} - \frac{1}{(1-r)^2}\right]$$

$$i.e \quad 1-|\phi(z)| \ge 1 + \frac{-2r+r^2}{(1-r)^2} + \frac{(1-r)(n+1)r^n + r^{n+1}}{(1-r)^4} - \frac{1}{(1-r)^4} + \frac{1}{(1-r)^2}$$

$$= 1 + \frac{1-2r+r^2}{(1-r)^2} + \frac{(1-r)(n+1)r^n + r^{n+1}}{(1-r)^4} - \frac{1}{(1-r)^4}$$

$$= 1 + 1 + \frac{(1-r)(n+1)r^n + r^{n+1}}{(1-r)^4} - \frac{1}{(1-r)^4} > 2 + 0 - \frac{1}{(1-r)^4} = 2 - \frac{1}{(1-r)^4}$$

since $0 < r < 1$ $\quad i.e. \quad 0 < 1-r.$

Assume that

$$2 - \frac{1}{(1-r)^4} > 0 \quad \Rightarrow \quad 2 > \frac{1}{(1-r)^4} \quad \Rightarrow \quad (1-r)^4 > \frac{1}{2} \quad \Rightarrow \quad 1-r > \frac{1}{\sqrt[4]{2}}$$

$$\Rightarrow \quad 1 - \frac{1}{\sqrt[4]{2}} > r \quad \Rightarrow \quad r < 1 - \frac{1}{\sqrt[4]{2}} < 1; \quad \left(1 - \frac{1}{\sqrt[4]{2}} \cong 1 - 0.84 = 0.1591\right)$$

Thus we have

$$|\mathbf{h}'(z)|^2 \,\mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right] \ge 1 - |\phi(z)| > 2 - \frac{1}{(1-r)^4} > 0, \qquad \left(|z| = r < 1 - \frac{1}{\sqrt[4]{2}} < 1\right)$$

$$\Rightarrow \quad \mathbf{Re}\left[\frac{s_n'(z,\mathbf{f})}{\mathbf{h}'(z)}\right] > 0, \qquad \left(|z| = r < 1 - \frac{1}{\sqrt[4]{2}} < 1\right).$$

Hence $s_n(z,\mathbf{f})$ are close-to-cap like functions in $|z| < 1 - {}^{-4}\!\sqrt{2}.$ / /

Concluisision Entire conclusion of the 13 theorems discussed is presented in the following.

1] Any $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univalent function in the open disc $\{z/|z| < 3^{-1} < 1\}$ by the Theorem **2.**

2] Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is any univalent function in the disc $\{z/|z| < 1\}.\}$ Then

i) $\mathbf{f}(z)$ has the property $|a_k| \le 1$ $(k = 2,3,4,\cdots)$ by Theorem **1;**

ii) $\mathbf{f}(z)$ is cap like function in the disc $\{z/|z| < 6^{-1} < 1\}$ by Theorem **3.**

iii) $s_n(z,\mathbf{f}) = z + \sum_{k=2}^{n} a_k z^k$ is univalent in $\{z/|z| < 3^{-1}\}$, $n \in ¥ -\{1\}$ by Theorem **6.**

iv) $\mathbf{f}(z)$ is star like function in the open disc $\{z/|z| < 1 - 3^{-1}\sqrt{6} < 1\}$ by Theorem **10.**

3] If $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is univlent and cap like function, then $z\mathbf{f}'(z)$ is star like function in the disc $\{z/|z| < 1 - 3^{-1}\sqrt{6} < 1\}$ by Theorem **4.**

4] If $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is star like function of order $\alpha < 1$, then $(z+1)\mathbf{f}(z)$ is star like function of order $\alpha \geq 0.4$ by Theorem **5.**

5] If $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is star like function in $\{z/|z| < 1\}$, then $s_n(z,\mathbf{f})$ $(n = 2,3,4,\cdots)$ are star like functions in $|z| < 1 - \sqrt[4]{2}$ by Theorem **7.**

6] Let $\mathbf{L}(z) = z(1-z)^{-1}$. Then $s_n(z,\mathbf{L})$ $(n = 2,3,4,\cdots)$ is cap like function in disc $|z| < 0.25$ by Theorem **8.**

7] Convolution of any two univalent functions $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ defined as $(\mathbf{f} * \mathbf{g})(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$ is univlent function in the open disc $\{z/|z| < 3^{-1}\}$, and is cap like functionin the open disc $\{z/|z| < 6^{-1}\}$ by Theorem **9.**

8] Let $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is any close-to-cap like function in $\{z/|z| < 1\}$. Then

i) $\mathbf{f}(z)$ is univalnt function by Theorem **11.**

ii) $s_n(z,\mathbf{f})$ are close-to-cap like functions in $|z| < 1 - \sqrt[4]{2}$ by Theorem **13.**

9] Let $\mathbf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is close-to-cap like function, $\mathbf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then $(\mathbf{f} * \mathbf{g})(z)$ is close-to-cap like function provided $|(\mathbf{f} * \mathbf{g})'(z) - \mathbf{g}'(z) + 1| \geq 1$ by Theorem **12.**

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