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Analytic Analysis for Oil Recovery During Cocurrent Imbibition in Inclined Homogeneous Porous Medium

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Abstract—This paper focuses on the analysis of cocurrent imbibition phenomenon which occurs during secondary oil recovery process.In cocurrent imbibition, a strongly wetting phase(water) displaces a non-wetting phase(oil) spontaneously under the influence of capillary forces such that the oil moves in the same direction to the water. We use an optimal homotopy analysis method to derive an approximate analytical expression for saturation of water when the viscosity of the non-wetting phase is non-negligible.

Keywords-*Cocurrent imbibition; Darcy law; Mass balance equation; Optimal Homotopy analysis method*.

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I. Introduction

When a porous medium is partially covered by water, oil recovery is dominated by cocurrent imbibition i.e. the production of non-wetting phase has the same direction of flow as the wetting phase.This phenomenon has been studied by many researchers and solved by different methods [1-5]. To find the distribution of water saturation in porous medium, pertaining nonlinear partial differential equation(PDE) should be solved with appropriate conditions. The purpose of this work is to solve nonlinear PDE describing cocurrent imbibition in inclined homogeneous porous medium by Optimal Homotopy analysis method(OHAM).This method has been applied recently to a number of problems for solving nonlinear ordinary and partial differential equations[15-26].

During secondary oil recovery process, it is assumed that the water is injected into fractured oil saturated inclinedhomogeneous porous medium and cocurrent imbibition occurs. It is also assumed that the macroscopic behavior of fingers is governed by statistical treatment. Thus, only average cross- sectional area occupied by fingers is taken into account, the size and shape of individual fingers are ignored. The velocities of both the phases are considered under gravitational and inclination effect. We assume that the porosity and the permeability of the porous medium are constant for the investigated flow system. The saturation of injected water $S_i(x,t)$ is then defined as the average cross-sectional area occupied by injected water at distance x and time t.

II. Governing Equations

The generalized Darcy's law for the wetting and non-wetting phases[7]:

$$V_i = -\frac{k_i}{\mu_i} K \left[\frac{\partial p_i}{\partial x} + \rho_i g sin\theta\right] \qquad (1)$$

$$V_n = -\frac{k_n}{\mu_n} K \left[\frac{\partial p_n}{\partial x} + \rho_n g sin\theta\right] \qquad (2)$$

where V_i and V_n are the velocities of water and oil respectively, k_i and k_n are the relative permeabilities of water and oilrespectively, μ_i and μ_n are the constant viscosities of water and oil respectively, K is the permeability of the inclined homogeneous porous medium, p_i and p_n are the pressures of water and oil respectively, ρ_i and ρ_n are the constant densities of water and oil respectively, g is the acceleration due to gravity, θ is the angle of inclination with porous matrix.

Mass balance of water volume assuming incompressible flow in one dimension with no overall flow can be expressed as follows [7-8]:

$$P\frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \qquad (3)$$

where P is the porosity of the medium.

The expression of the total velocity V_t of two phases in cocurrent imbibition phenomenon can be written as [11]

$$V_t = V_i + V_n \qquad (4)$$

The relation between the capillary pressure (p_c) generated by an interface and the difference in pressure across the interface between the non-wetting and wetting phases is [9]:

$$p_c = p_n - p_i \qquad (5)$$

We assume the linear relationship between capillary pressure and phase saturation as [12]

$$p_c(S_i) = -\beta S_i \qquad (6)$$

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where β is a constant.

According to Scheidegger and Johnson [13], we consider the analytical relationship between relative permeability and phase saturation as

$$k_i = S_i \tag{7}$$

The pressure of injected water can be expressed in the form

$$p_i = \frac{p_n + p_i}{2} + \frac{p_i - p_n}{2} = \bar{p} - \frac{1}{2}p_c \tag{8}$$

where $\bar{p}$ is the average pressure which is constant.

Using (1) to (8), we obtain the following nonlinear partial differential equation for the saturation of injected phase:

$$P\frac{\partial S_i}{\partial t} - \frac{K\beta}{2\mu_i}\frac{\partial}{\partial x}\left[S_i\frac{\partial S_i}{\partial x}\right] - \frac{K\rho_i g sin\theta}{\mu_i}\frac{\partial S_i}{\partial x} = 0 \tag{9}$$

Using dimensionless variables

$$X = \frac{x}{L}, \qquad T = \frac{K\beta t}{2\mu_i L^2 P}$$

(9) reduces to

$$\frac{\partial S_i}{\partial T} = \frac{\partial}{\partial X}\left[S_i\frac{\partial S_i}{\partial X}\right] + A\frac{\partial S_i}{\partial X} \tag{10}$$

where $A = \frac{2L\rho_i gsin\,\theta}{\beta}$ and $S_i(x,t) = S_i(X,T)$.

Eq. (10) is the nonlinear partial differential equation governing cocurrent imbibition phenomenon in inclined homogeneous porous medium. The solution of this equation represents the water saturation.

We assume following boundary conditions for the saturation of injected phase:

$$S_i(0,T) = aT \quad for\, T > 0 \tag{11}$$

and

$$S_i(1,T) = bT \quad for\, T > 0 \tag{12}$$

where a and b are constants.

We solve (10) together with boundary conditions (11) and (12) using optimal homotopy analysis method to obtain saturation distribution of water.

III. OPTIMAL HOMOTOPY ANALYSIS SOLUTION

To solve(10) by optimal homotopy-analysis method, we choose the initial approximation

$$S_{i_0}(X,T) = aT + \frac{(b-a)(1-e^{-X})}{(1-e^{-1})}T \tag{13}$$

of $S_i(X,T)$ which satisfies boundary conditions and the auxiliary linear operator as

$$\mathcal{L}[\varphi(X,T;q)] = \frac{\partial^2\varphi(X,T;q)}{\partial X^2} + \frac{\partial\varphi(X,T;q)}{\partial X} \tag{14}$$

with the property $\mathcal{L}[C] = 0$, where C is integral constant and $\varphi(X,T;q)$ is an unknown function.Furthermore, in the view of (10), we have defined the nonlinear operator as

$$\mathcal{N}[\varphi(X,T;q)] = \varphi(X,T;q)\frac{\partial^2\varphi(X,T;q)}{\partial X^2} + \left\{\frac{\partial\varphi(X,T;q)}{\partial X}\right\}^2 + A\frac{\partial\varphi(X,T;q)}{\partial X} - \frac{\partial\varphi(X,T;q)}{\partial T} \tag{15}$$

By means of the optimal homotopy analysis-method, Liao [20] constructs the so-called zeroth-order deformation equation

$$(1-q)\mathcal{L}\left[\varphi(X,T;q) - S_{i_0}(X,T)\right] = c_0 q H(X,T)\mathcal{N}[\varphi(X,T;q)] \tag{16}$$

where $q \in [0,1]$ is the embedding parameter, $c_0 \neq 0$ is convergence control parameter and $H(X,T)$ is nonzero auxiliary function.

It is obvious that for the embedding parameter $q = 0$ and $q = 1$, (16) becomes

$$\varphi(X,T;0) = S_{i_0}(X,T) \tag{17}$$

and

$$\varphi(X,T;1) = S_i(X,T) \tag{18}$$

respectively. Thus, as q increases from 0 to 1, the solution $\varphi(X,T;q)$ varies from the initial guess $S_{i_0}(X,T)$ to the solution $S_i(X,T)$ of (10).

Obviously, $\varphi(X,T;q)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $S_{i_0}(X,T)$ and the auxiliary parameter c_0. We have great freedom to select all of them.

Assuming that all of them are so properly chosen that the Taylor series

$$\varphi(X,T;q) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T)q^m \tag{19}$$

exists and converges at $q = 1$, we have the homotopy-series solution

$$S_i(X,T) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T) \tag{20}$$

where

$$S_{i_m}(X,T) = \frac{1}{m!}\frac{\partial^m \varphi(X,T;q)}{\partial q^m}\bigg|_{q=0} \quad (21)$$

Differentiating the zeroth order deformation equation (16) m times with respect to the embedding parameter q and then dividing by $m!$ and finally setting $q = 0$, we have the so called high order deformation equation

$$\mathcal{L}[S_{i_m}(X,T) - \chi_m S_{i_{m-1}}(X,T)] = c_0 H(X,T)\mathcal{R}_m(X,T) \quad (22)$$

subject to the boundary conditions

$$S_{i_m}(0,T) = 0 \text{ and} S_{i_m}(1,T) = 0, \qquad m \geq 1 \qquad (23)$$

where

$$\mathcal{R}_m(X,T) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\varphi(X,T;q)]}{\partial q^{m-1}}\bigg|_{q=0}, m \geq 1 \quad (24)$$

$$\text{and } \chi_m = \begin{cases} 0, & if\ m \leq 1 \\ 1, & if\ m > 1. \end{cases} (25)$$

For simplicity, assume $H(X,T) = 1$. The eqs. (22) are second order ordinary linear differential equations for all $m \geq 1$ and can be solved by symbolic computation software such as Mathematica. Thus we convert the original nonlinear problem (10)-(11)-(12) into an infinite sequence of linear sub-problems governed by (22)-(23).

Hence the approximate analytical solution to the given nonlinear problem takes the following form:

$$\begin{aligned} S_i(X,T) &= aT + \alpha(1-e^{-X})T \\ &+ c_0\left\{\frac{1-e^{-X}}{1-e^{-1}}(\alpha a e^{-1}T^2 + \alpha^2 e^{-2}T^2\right. \\ &- \alpha e^{-1}AT + \alpha e^{-1} + a + \alpha) + \alpha^2 e^{-X}T^2 \\ &- \alpha^2 e^{-X}XT^2 - a\alpha e^{-X}XT^2 - \alpha^2 e^{-2X}T^2 \\ &\left. + \alpha A e^{-X}XT - \alpha e^{-X}X - aX - \alpha X\right\} \\ &+ \cdots \end{aligned} \quad (26)$$

where $\alpha = \frac{b-a}{1-e^{-1}}$.

The solution represents the saturation distribution of water $S_i(X,T)$ at distance X and time T. The convergence of the solution depends on the convergence control parameter c_0. As shown in [14, 17, 20], we can determine the possible optimal value of convergence-control parameter c_0 by minimizing the averaged squared residual

$$E_m = \frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathcal{N}\left[\sum_{n=0}^{m} S_{i_n}\left(\frac{i}{M},\frac{j}{N}\right)\right]\right\}^2 \quad (27)$$

where we have chosen $M = N = 50$ in this paper.

At the given order of approximation, the minimum of the averaged squared residual corresponds to the optimal approximation.

The value of c_0 can be optimally identified from the condition

$$\frac{dE_m(c_0)}{dc_0} = 0 \qquad (28)$$

In this paper, the command NMinimize of the computer algebra system Mathematica is used to find out the minimum of averaged squared residual and the corresponding optimal convergence-control parameter.

IV. Numerical Results and Discussion

To obtain the numerical values of the solution, we assume the value of constants as $L = 1; \rho_i = 0.1;\ g = 9.8; \beta = 2;\ a = 0.001; b = 0.01; \alpha = 0.01;\ A = 0$ for $\theta = 0^\circ; A = 0.09$ for $\theta = 5^\circ; A = 0.17$ for $\theta = 10^\circ$.

A. $\theta = 0^\circ$ inclination with porous matrix.

Fig.1 shows the curve of averaged squared residual at the 10th order of approximation E_{10} versus c_0 when $\theta = 0^\circ$. Using Mathematica, we find that E_{10} has its minimum value 7.97134×10^{-6} at $c_0 = -0.83$ which can be seen in Fig. 1.

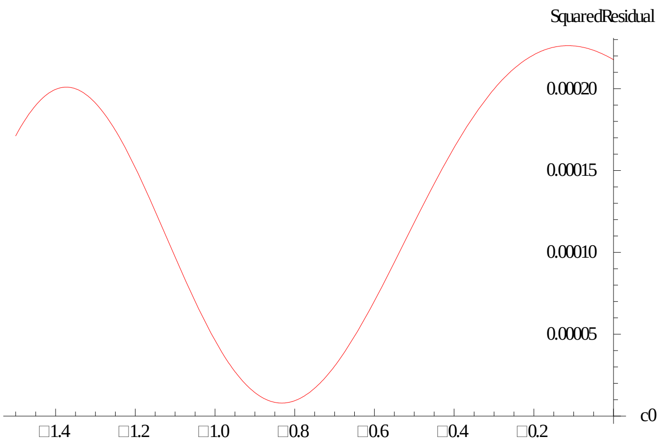

Fig.1. Averaged Squared Residual E_{10} when $\theta = 0^\circ$.

Table 1 indicates the numerical values of saturation of injected water up to 10th order approximation when $\theta = 0^\circ$ using $c_0 = -0.83$. The graph of saturation of injected water versus distance X for fixed time $T = 10, 20, \ldots, 100$ is shown in Fig. 2.

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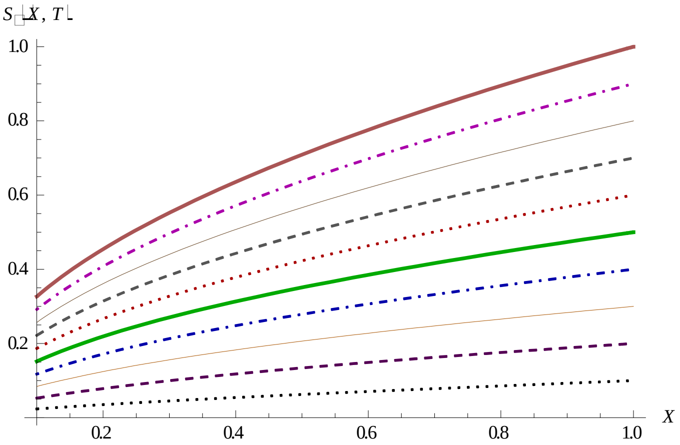

Fig.2: Saturation of water versus distance X for fixed time $T = 10, 20, \ldots, 100$ when $\theta = 0^{\circ}$.

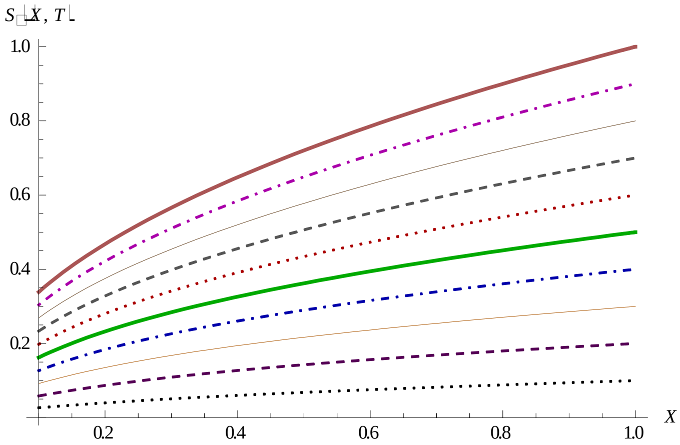

Fig.4: Saturation of water versus distance X for fixed time $T = 10, 20, \ldots, 100$ when $\theta = 5^{\circ}$.

B. $\theta = 5^{\circ}$ inclination with porous matrix.

Fig.3 shows the curve of averaged squared residual at the 10th order of approximation E_{10} versus c_0 when $\theta = 5^{\circ}$. Using Mathematica, we find that E_{10} has its minimum value 2.45049×10^{-5} at $c_0 = -0.84$ which can be seen in Fig. 3 also.

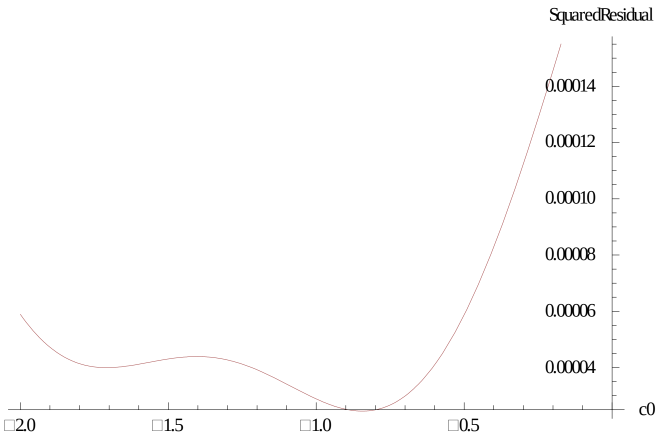

Fig.3. Averaged Squared Residual E_{10} when $\theta = 5^{\circ}$.

The numerical values of saturation of injected water up to 10th order approximation are obtained when $\theta = 5^{\circ}$ using $c_0 = -0.84$ (Table 2). Fig.4 shows the graph of saturation of injected water versus distance X for fixed time $T = 10, 20, \ldots, 100$.

C. $\theta = 10^{\circ}$ inclination with porous matrix.

Fig.5 shows the curve of averaged squared residual at the 10th order of approximation E_{10} versus c_0 when $\theta = 10^{\circ}$. Using Mathematica, we find that E_{10} has its minimum value 6.68927×10^{-5} at $c_0 = -0.57$ which can be seen in Fig. 5 also.

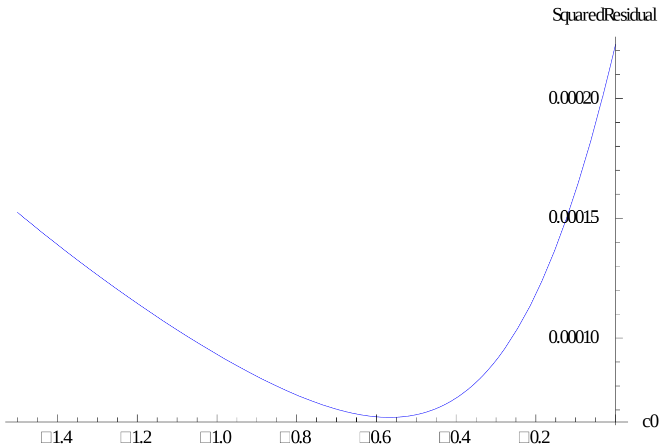

Fig.5: Averaged Squared Residual E_{10} when $\theta = 10^{\circ}$.

Table 3 indicates the numerical values of saturation of injected water up to 10th order approximation for $\theta = 10^{\circ}$ taking $c_0 = -0.57$. The graph of saturation of injected water versus distance X for fixed time $T = 10, 20, \ldots, 100$ is shown in Fig.6.

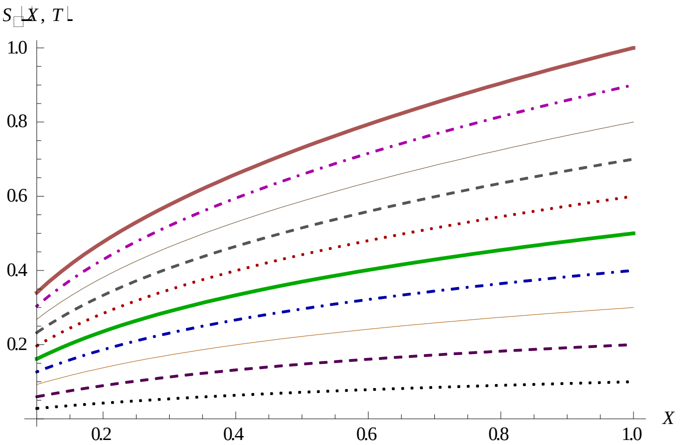

Fig.5: Saturation of water versus distance X for fixed time $T = 10, 20, \ldots, 100$ when $\theta = 10^{\circ}$.

V. CONCLUSIONS

The approximate analytical solution is obtained for the concurrent imbibition phenomenon in inclined homogeneous porous medium by optimal homotopy analysis method. The convergence of solution is guaranteed by using optimal value of convergence control parameter. The saturation of injected water increases when angle of inclination with porous matrix increases. The water saturation increases when the distance increases for fixed time.

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TABLE 1: Numerical values of the saturation of injected water for $\theta = 0^{\circ}$.

T	X=0.1	X=0.2	X=0.3	X=0.4	X=0.5	X=0.6	X=0.7	X=0.8	X=0.9	X=1
10	0.023558	0.052737	0.084317	0.117539	0.151809	0.186671	0.221786	0.256907	0.291870	0.326570
20	0.035053	0.078396	0.124219	0.171351	0.219011	0.266713	0.314182	0.361291	0.408013	0.454380
30	0.045124	0.099590	0.155979	0.213130	0.270387	0.327424	0.384118	0.440467	0.496523	0.552360
40	0.054205	0.117842	0.182779	0.248077	0.313282	0.378216	0.442855	0.507238	0.571428	0.635482
50	0.062597	0.134078	0.206341	0.278724	0.350948	0.422933	0.494692	0.566269	0.637711	0.709059
60	0.070514	0.148886	0.227660	0.306426	0.385033	0.463450	0.541702	0.619823	0.697846	0.775797
70	0.078112	0.162653	0.247346	0.331977	0.416475	0.500835	0.585078	0.669227	0.753306	0.837332
80	0.085506	0.175644	0.265796	0.355878	0.445867	0.535765	0.625587	0.715349	0.805063	0.894743
90	0.092781	0.188048	0.283283	0.378472	0.473612	0.568708	0.663765	0.758792	0.853796	0.948781
100	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

TABLE 2: Numerical values of the saturation of injected water for $\theta = 5^{\circ}$.

T	X=0.1	X=0.2	X=0.3	X=0.4	X=0.5	X=0.6	X=0.7	X=0.8	X=0.9	X=1
10	0.026667	0.058488	0.092117	0.126890	0.162294	0.197947	0.233571	0.268980	0.304060	0.338751
20	0.039914	0.086912	0.135186	0.183872	0.232434	0.280582	0.328194	0.375263	0.421848	0.468045
30	0.050800	0.109120	0.167810	0.226225	0.284086	0.341330	0.398020	0.454269	0.510202	0.565931
40	0.060037	0.127311	0.194233	0.260515	0.326129	.0391172	0.455784	0.520096	0.584214	0.648215
50	0.068108	0.142791	0.216691	0.289831	0.362348	0.434407	0.506151	0.577688	0.649093	0.720414
60	0.075353	0.156374	0.236440	0.315780	0.394605	0.473082	0.551332	0.629434	0.707439	0.785376
70	0.082012	0.168583	0.254236	0.339284	0.423942	0.508350	0.592599	0.676742	0.760813	0.844833
80	0.088260	0.179772	0.270561	0.360917	0.451012	0.540946	0.630776	0.720537	0.810249	0.899926
90	0.094224	0.190186	0.285739	0.381064	0.476219	0.571373	0.666437	0.761465	0.856468	0.951454
100	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1

TABLE 3: Numerical values of the saturation of injected water for $\theta = 10^{\circ}$.

T	X=0.1	X=0.2	X=0.3	X=0.4	X=0.5	X=0.6	X=0.7	X=0.8	X=0.9	X=1
10	0.028016	0.059594	0.092584	0.126638	0.161460	0.196804	0.232462	0.268268	0.304086	0.339813
20	0.042326	0.089470	0.137675	0.186452	0.235438	0.284372	0.333078	0.381446	0.429419	0.476979
30	0.053926	0.112824	0.172021	0.231135	0.289934	0.348293	0.406165	0.463555	0.520504	0.577070
40	0.063524	0.131669	0.199433	0.266655	0.333284	0.399338	0.464878	0.529980	0.594728	0.659202
50	0.071631	0.147308	0.222156	0.296233	0.369638	0.442485	0.514886	0.586944	0.658745	0.730359
60	0.078621	0.160608	0.241569	0.321716	0.401235	0.480281	0.558978	0.637423	0.715688	0.793825
70	0.084769	0.172164	0.258556	0.344224	0.429378	0.514171	0.598712	0.683078	0.767322	0.851480
80	0.090283	0.182395	0.273705	0.364474	0.454883	0.545050	0.635055	0.724951	0.814770	0.904535
90	0.095320	0.191600	0.287423	0.382953	0.478295	0.573518	0.668662	0.763753	0.858808	0.953836
100	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0