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Essential T-Goldie*-Supplemented Modules

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Abstract—. In our work, we defined Essential T- $\boldsymbol{\beta}^*$-equivalent is relation between two submodules of a module N. A submodules K and H of a modules N are Essential- T-$\boldsymbol{\beta}^*$ equivalent (K $\boldsymbol{\beta}^*_{ET}$ H), if $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$, for some $\boldsymbol{X \ll_{ET} N}$. We give many characterizations of the relation $\boldsymbol{\beta}^*_{ET}$ equivalent. So, we defined the Essential T- Goldie* lifting Modules and we give many properties related with this type. Keyword: ET-small submodule, Essential T-lifting module, Essential T-Goldie*lifting modules

Keywords - ET-small submodule, essential T-weak supplemented, essential T-H-supplemented, Essential T-Goldie*lifting modules.

I. Introduction

Throughout this paper rings are associative with identity and modules are unital left R-modules. Let A $\leq$ M, then A is called "Essential T-small submodule of M", if $\forall$ N$\leq_e$M such that T⊆A+N, hence T ⊆ N [1].. Recalled M is called an essential T-hollow(ET-hollow) module if for every proper submodule H of M is an ET-small submodule of M[2]. Recalled M is an ET-hollow-lifting module; Let T be a submodule of a module M, if for every submodule H of M with $\frac{M}{H}$ is E($\frac{T+H}{H}$)-hollow, then there exists; a direct summand D of M such that D $\subseteq_{E(T)ce}$H [3,4]. Recalled that N is called Essential T-H-supplemented module if for every submodule Y of N, there exists B $\leq_\oplus$N, such that T⊆Y+C if and only if T⊆B+C, $\forall$ C $\leq_e$ N[5]. A module N is named Essential T-lifting module, if $\forall$ H $\leq$ N, there subsist Y$\leq_\oplus$N and V$\ll_{ET}$N whereas H=Y+V[6]. Recalled that β^*_T -equivalent relation, let T $\leq$ U, then S and L of U are β^*_T -equivalent (S β^*_T L) if $\frac{S+L}{S} \leq \frac{S+N}{S}$ and $\frac{S+L}{L} \leq \frac{L+N}{L}$, for some N$\ll_T$U. A module U is named a Goldie*-T-lifting module, where $\forall K \leq U$, there subsist C$\leq_\oplus$U whereas K β^*_T C . The essential T- β^* relation. Let V, W and T are submodules of M, hence V and W are said to be essential-T-β^* equivalent (V β^*_{ET}W), if there exists K is Essential T-small sub-module of M whereas $\frac{S+W}{S} \leq \frac{S+K}{S}$ and $\frac{S+W}{W} \leq \frac{W+K}{W}$ [7,8] . We study Essential T- β^* relation on the set of sub-modules of a module M .Also we define the Essential essential T-Goldie*-supplemented modules and we give some properties and characterizations for this type module.

The $\boldsymbol{\beta}^*_{ET}$- Relation. In this section, we develop the basic properties of the $\boldsymbol{\beta}^*_{ET}$ on the set of submodules of M.
Ease of Use

Definition (2.1): Let T $\leq$ M .We say submodules X , Y of M are β^*_{ET}- equivalent (X β^*_{ET} Y) if $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$.

Example (2.2):

1. Consider the module Z_4 as Z-module. Let T = Z_4, X = $\{\bar{0},\bar{2}\}$ and Y = $\{\bar{0}\}$. Then $\frac{X+Y}{X}$ = 0 and $\frac{X+Y}{Y} \cong \{\bar{0},\bar{2}\}$. Hence 0 = $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X}) \cong \{\bar{0},\bar{2}\}} \frac{M}{X} \cong \{\bar{0},\bar{2}\}$ and $\{\bar{0},\bar{2}\} \cong \frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y}) \cong Z4} \frac{M}{Y} \cong Z_4$ .Thus X is β^*_{ET}- equivalent to Y.

2. Consider the module Z_6 as Z-module .Let T = $\{\bar{0},\bar{2},\bar{4}\}$, X = $\{\bar{0},\bar{3}\}$ and Y = $\{\bar{0},\bar{2},\bar{4}\}$ .Then $\frac{M}{X} = \frac{X+Y}{X} = \frac{T+X}{X} \cong \{\bar{0},\bar{2},\bar{4}\}$ .But $\frac{X+Y}{X}$ is not $\frac{T+X}{X}$ -small in $\frac{M}{X}$ .Therefore X is not β^*_{ET}-equivalent to Y.

Proposition (2.3) β^*_{ET} is an equivalence relation .

Proof: Clearly that β^*_{ET} is reflexive and symmetric .To show that β^*_T is transitive .Let X,Y and Z be submodules of M such that X β^*_{ET} Y and Y β^*_{ET} Z .we want to show that X β^*_{ET} Z .Since X β^*_{ET} Y and Yβ^*_{ET}Z .Then $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$, $\frac{Y+Z}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$ and $\frac{Y+Z}{Z} \ll_{E(\frac{T+Z}{Z})} \frac{M}{Z}$. Let $\frac{K}{X}$ be an essential submodule of $\frac{M}{X}$ such that $\frac{T+X}{X} \subseteq \frac{X+Z}{X} + \frac{K}{X}$ and hence T⊆T+X⊆Z+K .Therefore $\frac{T+Y}{Y} \subseteq \frac{Z+Y}{Y} + \frac{K+Y}{Y}$ .Since $\frac{Y+Z}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$,then $\frac{T+Y}{Y} \subseteq \frac{K+Y}{Y}$ and hence T⊆T+Y⊆Y+K. Therefore $\frac{T+X}{X} \subseteq \frac{Y+X}{X} + \frac{K}{X}$ .Since $\frac{X+Y}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$, then $\frac{T+X}{X} \subseteq \frac{K}{X}$ .Thus $\frac{X+Z}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ .Now, let $\frac{K}{Z}$ be anessential submodule of $\frac{M}{Z}$ such that $\frac{T+Z}{Z} \subseteq \frac{X+Z}{Z} + \frac{K}{Z}$ and hence T⊆T+Z⊆X+K .Therefore

$\frac{T+Y}{Y} \subseteq \frac{X+Y}{Y} + \frac{K+Y}{Y}$ .Since $\frac{X+Y}{Y} \ll_{\frac{T+Y}{Y}} \frac{M}{Y}$,then $\frac{T+Y}{Y} \subseteq \frac{K+Y}{Y}$ and hence T⊆T+Y⊆Y+K . Therefore $\frac{T+Z}{Z} \subseteq \frac{Y+Z}{Z} + \frac{K}{Z}$ .Since $\frac{Y+Z}{Z} \ll_{E(\frac{T+Z}{Z})} \frac{M}{Z}$, then $\frac{T+Z}{Z} \subseteq \frac{K}{Z}$ .Therefore $\frac{X+Z}{Z} \ll_{E(\frac{T+Z}{Z})} \frac{M}{Z}$, so X δ_T^* Z .Thus β_{ET}^* is an equivalence relation.

Theorem (2.4): Let X, Y be submodules of a module M .Then the following statement are

1. X β_{ET}^* Y.

2. For each submodule K of M such that T⊆X+Y+K, then T⊆X+K and T⊆Y+K.

3. If K is a submodule of M with T⊆X+K, then T⊆Y+K and if H is submodule of M with T⊆Y+H, then T⊆X+H.

Proof: **1⇒2**

Let X β_{ET}^* Y, then $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$ .Let K be an essential submodule of M such that T⊆X+Y+K .Then $\frac{T+Y}{Y} \subseteq \frac{X+Y}{Y} + \frac{K+Y}{Y}$ .Since $\frac{X+Y}{Y} \ll_{\frac{T+Y}{Y}} \frac{M}{Y}$, then $\frac{T+Y}{Y} \subseteq \frac{K+Y}{Y}$ and hence T⊆T+Y⊆Y+K. Therefore $\frac{T+X}{X} \subseteq \frac{Y+X}{X} + \frac{K+X}{X}$. Since $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$, then $\frac{T+X}{X} \subseteq \frac{K+X}{X}$ and hence T⊆T+X⊆K+X .Thus T⊆K+X.

2⇒1 Let $\frac{K}{Y}$ be an essential submodule of $\frac{M}{Y}$ such that $\frac{T+Y}{Y} \subseteq \frac{X+Y}{Y} + \frac{K}{Y}$ .Then T⊆X+Y+K. By (**2**), T⊆Y+K.But Y ≤ K, therefore T⊆K. Hence $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$ .Now, let $\frac{K}{X}$ be a submodule of $\frac{M}{X}$ such that $\frac{T+X}{X} \subseteq \frac{X+Y}{X} + \frac{K}{X}$ .Then T⊆X+Y+K. By (**2**), T⊆X+K. But X ≤ K, therefore T⊆K. Hence $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$. Thus X β_{ET}^* Y.

2⇒3 Let K be an essential submodule of M such that T⊆X+K .Then T⊆X+Y+K .By (**2**), T⊆Y+K .Now, let H be submodule of M such that T⊆Y+H .Then T⊆X+Y+H .By (**2**), T⊆X+H .

3⇒2 Let K be submodule of M and assume T⊆X+Y+K .By (**3**), T⊆Y+(Y+K) and T⊆X+(X+K) and hence T⊆Y+K and T⊆X+K .

3⇒1 Let $\frac{K}{X}$ be an essential submodule of $\frac{M}{X}$ such that $\frac{T+X}{X} \subseteq \frac{X+Y}{X} + \frac{K}{X}$ .Then T⊆T+X⊆ X+Y+K⊆Y+K .By (**3**), T⊆X+K .Since X ≤ K, then T⊆K and hence $\frac{T+X}{X} \subseteq \frac{K}{X}$ .Therefore $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ .Now, let $\frac{K}{Y}$ be a submodule of $\frac{M}{Y}$ such that $\frac{T+Y}{Y} \subseteq \frac{X+Y}{Y} + \frac{K}{Y}$ .Then T⊆T+Y⊆X+Y+K⊆X+K .By (**3**), T⊆Y+K .Since Y ≤ K, then T⊆K and hence $\frac{T+Y}{Y} \subseteq \frac{K}{Y}$ .Therefore $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$ .Thus X β_{ET}^*Y.

Proposition (2.5): Let X be submodule of a module M. Then X β_{ET}^*0 if and only if X$\ll_T$M.

Proof: Suppose that X β_{ET}^* 0, then $\frac{X+0}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and hence $0 \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+0}{0} \ll_{E(\frac{T+0}{0})} \frac{M}{0}$. Therefore X$\ll_{ET}$ M .Conversely, assume that X$\ll_{ET}$ M .We want to show that X β_{ET}^* 0 . Then $\frac{X+0}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and hence $0 \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ .Now $\frac{X+0}{0} \ll_{E(\frac{T+0}{0})} \frac{M}{0}$ .Hence X$\ll_{ET}$M . Thus X β_{ET}^* 0.

Proposition (2.6) Let X and Y be submodules of a module M such that X=Y+B and Y=X+A, where A,B $\ll_{ET}$ M .Then X β_{ET}^* Y.

Proof Let X=Y+B and Y=X+A, where A, B $\ll_T$M .Then X+Y=X+X+A=X+A and X+Y=Y+B+Y=Y+B .By proposition(2.3), X β_{ET}^* X .Since A$\ll_T$M ,then A β_{ET}^* 0 ,by proposition (2.5) and hence X+A β_{ET}^* X+0 .Since X+Y=X+A ,then X+Y β_{ET}^* X .By the same way then Y β_{ET}^* Y. Since B$\ll_{ET}$M ,then B β_{ET}^*0, by proposition(2.5) and hence (Y+B) β_{ET}^*(Y+0) .Since X+Y=Y+B, then (X+Y) β_{ET}^* Y .By proposition(2.3), then X β_{ET}^*Y.

Theorem (2.7) Let X, Y be submodules of a distributive module M such that X β_{ET}^*Y .Then:

1. X$\ll_{ET}$ M if and only if Y$\ll_{ET}$ M.

2. X has an essential T-weak supplemented K in M if and only if K is an essential T-weak supplemented of Y in M.

Proof(1): Assume that X$\ll_{ET}$ M. Let K be a submodule of M such that T⊆Y+K. Then T⊆X+Y+K .But X β_{ET}^*Y, therefore T⊆X+K, by Theorem(2.4). Since X$\ll_{ET}$M, then T⊆K. Thus Y$\ll_{ET}$M.

Conversely, is true by Proposition(2.3).

Proof: (**2**) Suppose that K is an essential T-weak supplemented of X in M .Then T⊆X+K and (X⋂K) $\ll_{ET}$ M .To show that K is an ET-weak supplemented of Y in M .Since X β_{ET}^*Y, then T⊆Y+K by Theorem(2.4). Now, to show that $(Y \cap K) \ll_{ET} M$ .Let N ≤ M such that T⊆(Y⋂K)+N .Then T⊆Y+N and T⊆K+N .Since X β_{ET}^* Y, then T⊆X+N by (2.4) .Then T⊆(X+N)⋂(K+N)⊆ N+(X⋂(K+N)), by modular Law. Since M is distributive module, then T⊆N+(X⋂K)+(X⋂N)⊆N+(X⋂K) .Since $X \cap K \ll_{ET} M$,then T⊆N .Therefore $Y \cap K \ll_{ET} M$ .Thus K is an essential T-weak supplemented of Y in M .Conversely, is true by Proposition(2.3).

Proposition (2.8):

Let M, N be modules and let $f : M \to N$ be an epimorphisim ,then :

1. If X, Y are submodules of M such that X β_{ET}^* Y, then f(X) $\beta_{Ef(T)}^*$ f(Y).

2. If X, Y are submodules of N such that X $\beta_{Ef(T)}^*$ Y, then $f^{-1}(X)\ \beta_{ET}^*\ f^{-1}(Y)$.

3. If f is a ET-small cover, X be submodule of M and K be submodule of N Such that $f(X)\ \beta_{Ef(T)}^* K$, then $X\ \beta_{ET}^*\ f^{-1}(K)$.

Proof: (**1**)

Suppose that X β_{ET}^* Y. Then $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$. To show that f(X) $\beta_{Ef(T)}^*$(Y). Let $\frac{K}{f\,(X)} \leq_e \frac{N}{f\,(X)}$ such that $\frac{f(T)+f(X)}{f(X)} \subseteq \frac{f(X)+f(Y)}{f(X)} + \frac{K}{f\,(X)}$. Then $f(T) + f(X) \subseteq f(X) + f(Y)$ + K and hence f (T+X) ⊆ f (X+Y) + f (w) , where f (w) = K .So f^{-1}(f (T+X)) ⊆ f^{-1}(f (X+Y) + f (w)), therefore T+X+ $Ker\,f$⊆X+Y+W+ $Ker\,f$. Then T⊆X+Y+W and hence $\frac{T+X}{X} \subseteq \frac{X+Y}{X} + \frac{W+X}{X}$. Since $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$, then $\frac{T+X}{X} \subseteq \frac{W+X}{X}$ and hence T+X⊆W+X. So f (T+X)⊆f (W+X), therefore $\frac{f(T)+f(X)}{f(X)} \subseteq \frac{K}{f\,(X)}$. Thus $\frac{f(X)+f(Y)}{f(X)} \ll_{E(\frac{f(T)+f(X)}{f(X)})} \frac{N}{f\,(X)}$. By the same way $\frac{f(X)+f(Y)}{f(Y)} \ll_{E(\frac{f(T)+f(Y)}{f(Y)})} \frac{N}{f\,(Y)}$. Thus f (X) $\beta_{Ef(T)}^*$ f (Y).

Proof: (**2**) Suppose that X $\beta^*_{Ef(T)}$Y. Then $\frac{X+Y}{X} \ll_{E\left(\frac{f(T)+X}{X}\right)} \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_{E\left(\frac{f(T)+Y}{Y}\right)} \frac{M}{Y}$. To show that f^{-1}(X) β^*_{ET} f^{-1}(Y). Let $\frac{K}{f^{-1}(X)} \leq_e \frac{M}{f^{-1}(X)}$ such that $\frac{T+f^{-1}(X)}{f^{-1}(X)} \subseteq \frac{f^{-1}(X)+f^{-1}(Y)}{f^{-1}(X)} + \frac{K}{f^{-1}(X)}$. Then T+ f^{-1} (X) ⊆ f^{-1} (X+Y) + f^{-1} (W), where f^{-1}(W) = K. So f (T+ f^{-11}(X)) ⊆f (f^{-1} (X+Y) + f^{-1}(W)) ,therefore $\frac{f(T)+X}{X} \subseteq \frac{X+Y}{X} + \frac{W+X}{X}$ .Since $\frac{X+Y}{X} \ll_{E\left(\frac{T+X}{X}\right)} \frac{M}{X}$,then $\frac{f(T)+X}{X} \subseteq \frac{W+X}{X}$ and hence f (T)+X⊆W+X .So f^{-1} (f(T)+X) ⊆ f^{-1}(W+X) ,therefore T+ f^{-1}(X) ⊆ f^{-1}(W) + f^{-1}(X). Then $\frac{T+f^{-1}(X)}{f^{-1}(X)} \subseteq \frac{K}{f^{-1}(X)}$. Thus $\frac{f^{-1}(X)+f^{-1}(Y)}{f^{-1}(X)} \ll_{E\left(\frac{T+f^{-1}(X)}{f^{-1}(X)}\right)} \frac{M}{f^{-1}(X)}$ .By the same way $\frac{f^{-1}(X)+f^{-1}(Y)}{f^{-1}(Y)} \ll_{E\left(\frac{T+f^{-1}(Y)}{f^{-1}(Y)}\right)} \frac{M}{f^{-1}(Y)}$. Thus f^{-1}(X) β^*_{ET} f^{-1}(Y).

Proof: (**3**) Suppose that f (X) $\beta^*_{Ef(T)}$ K .To show that X β^*_{ET} f^{-1}(K) .Let $\frac{H}{X} \leq_e \frac{M}{X}$ such that $\frac{T+X}{X} \subseteq \frac{X+f^{-1}(k)}{X} + \frac{H}{X}$ .Then T+X⊆X+ f^{-1}(X)+H and hence f (T+X) ⊆ f (X+f^{-1}(K)) + f (H). So $f(T) + f(X) \subseteq f(X)$ + K + f (H), therefore $\frac{f(T)+f(X)}{f(X)} \subseteq \frac{f(X)+K}{f(X)} + \frac{f(H)}{f(X)}$. Since f (X) $\beta^*_{Ef(T)}$K, then $\frac{f(T)+f(X)}{f(X)} \subseteq \frac{f(H)}{f(X)}$ and hence $f(T) + f(X) \subseteq f(H)$. Then f (T+X) ⊆ $f(H)$ and hence $T + X + Ker\,f \subseteq H + Ker\,f$. So $T \subseteq H + Ker\,f$. Since f is a T-small cover ,then T⊆H and hence $\frac{T+X}{X} \subseteq \frac{H}{X}$. Thus $\frac{X+f^{-1}(k)}{X} \ll_{E\left(\frac{T+X}{X}\right)} \frac{M}{X}$. Now, Let $\frac{H}{f^{-1}(k)} \leq \frac{M}{f^{-1}(k)}$ such that $\frac{T+f^{-1}(k)}{f^{-1}(k)} \subseteq \frac{X+f^{-1}(k)}{f^{-1}(k)} + \frac{H}{f^{-1}(k)}$. Then T+ f^{-1}(K)⊆X+ f^{-1} (K)+H and hence f(T+ f^{-1}(K)) ⊆ f(X+ f^{-1}(K))+ f(H). So $f(T) + K \subseteq f(X)$ + K + f (H), therefore $\frac{f(T)+K}{K} \subseteq \frac{f(X)+K}{K} + \frac{f\,(H)}{K}$. Since $f\,(X)\,\beta^*_{Ef(T)}K$, then $\frac{f(T)+K}{K} \subseteq \frac{f\,(H)}{K}$ and hence $f(T) + K \subseteq f(H)$. So f^{-1}(f(T)+K) ⊆f^{-1}(f(H)), therefore T+ f^{-1}(K)+ $Ker\,f$ ⊆H + $Ker\,f$ .Then T ⊆ H + $Ker\,f$. Since f is an ET-small cover, then T⊆H and hence $\frac{T+f^{-1}(k)}{f^{-1}(k)} \subseteq \frac{H}{f^{-1}(k)}$. So $\frac{X+f^{-1}(k)}{f^{-1}(k)} \ll_{E\left(\frac{T+f^{-1}(k)}{f^{-1}(k)}\right)} \frac{M}{f^{-1}(k)}$. Thus X β^*_{ET} f^{-1}(K).

Proposition (2.9): Let X_1, X_2, Y_1, Y_2 be submodules of a module M such that X_1 β^*_{ET} Y_1 and X_2 β^*_{ET} Y_2. Then (X_1+X_2) β^*_{ET} (Y_1+Y_2) and (X_1+Y_2) β^*_{ET} (Y_1+X_2) .

Proof: Let X_1 β^*_{ET} $Y_1$and X_2 β^*_{ET} Y_2 . To show that (X_1+X_2) β^*_{ET}(Y_1+Y_2). Let K be an essential submodule of M such that T⊆ ($X_1 + X_2$)+($Y_1 + Y_2$)+K, so T⊆ $X_1 + Y_1$ + (X_2+Y_2+K). Since X_1 $\beta^*_{ET}$$Y_1$, then T⊆ $X_1 + X_2$+Y_2 + K and T⊆ Y_1+ X_2+Y_2 + K. Since X_2 $\beta^*_{ET}$$Y_2$, then T ⊆ $X_1 + X_2$ + K and T ⊆ $Y_1 + Y_2$ +K . Thus ($X_1 + X_2$) β^*_{ET} ($Y_1 + Y_2$) ,by (2.4) .By the same way (X_1+Y_2) β^*_{ET} ($Y_1 + X_2$) .

Corollary (2.10): Let X, Y be submodules of a module M and $K \ll_{ET} M$ .Then X β^*_{ET} Y if and only if X β^*_{ET}(Y+K).

Proof: Suppose that X β^*_{ET}Y. Since $K \ll_{ET} M$, then K β^*_{ET} 0, by (2.5). Then X+0 β^*_{ET}Y+K, by (2.9). Thus X β^*_{ET} (Y+K).

Conversely, let X β^*_{ET} (Y+K), then $\frac{X+Y+K}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+Y+K}{Y+K} \ll_{E(\frac{T+Y+K}{Y+K})} \frac{M}{Y+K}$ .Since $\frac{X+Y}{X} \subseteq \frac{X+Y+K}{X}$, then $\frac{X+Y}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ .Since $\frac{X+Y}{Y} \subseteq \frac{X+Y+K}{Y+K}$, then $\frac{X+Y}{Y} \ll_{E(\frac{T+Y}{Y})} \frac{M}{Y}$,by (1.2.9) . Thus X β^*_{ET} Y.

Proposition (2.11): Let M be an essential T-hollow module[9] .Then X β^*_{ET}Y, for every proper submodules X and Y of M such that T⊄X and T⊄Y.

Proof: Let X ,Y be a proper submodules of M such that T⊄X and T⊄Y .Since M is hollow module, then $X \ll_{ET} M$ and $Y \ll_{ET} M$ .By (2.5) , X β^*_{ET}0 and Yβ^*_{ET}0 .Thus X β^*_{ET}Y, by (2.3).

3.Essential T-Goldie*-supplemented modules

DEFINITION 3.1. We say M is essential T-Goldie∗-supplemented if and only if for each $X \leq M$, there exists a supplement submodule S of M such that $X\,\beta^*_{ET}\,S$.

THEOREM 3.2. *M is $ET - G^*$-supplemented if and only if for each X ≤ M there exists a supplement S and a small submodule H of M such That X + H =S+H =X+S .*

Proof. Assume that M is G∗-supplemented. There exists a supplement S such that $X\,\beta^*_{ET}\,S$. Hence there exists $W \leq M$ such that S+W=M and $(S \cap W) \ll S$. By Proposition (2.9), $X\,\beta^*_{ET}(X+S)$ *. and* $S\,\beta^*_{ET}(X+S)$ From Theorem (2.7) W is an essential weak supplement for S, X , and X+S By the modular law, $X + H = S + H = X + S$, where $H = (X + S) \cap W \ll M$.

The converse follows from (2.4).

COROLLARY(3.3). *(i) If for each $X \leq M$ there exists a supplement S and $H \ll M$ such that $X = S + H$, then M is $ET - G^* -$supplemented. The converse holds if M is also distributive.*

(ii) Let M be G∗-supplemented such that Rad(M) ≤ X. Then X =S + H, where S is a supplement and H ≪ M.

Proof. (i) From Theorem (3.2) the hypothesis implies that M is G∗-supplemented. Assume that M *is* $ET - G^* -$*supplemented and distributive. Let* $X \leq M$ *.Then there are* $S, L \leq M$ *such that* $X\beta^*_{ET}S$, $S + L = M$ *and* $S \cap L \ll S$. *By Theorem (2.3),* $X + L = M$. *So* $S = S \cap (X + L) = S \cap X + S \cap L = S \cap X$. *Hence* $S \leq X$. *From Theorem (2.6), L is an ET-weak supplement of X, so* $X \cap L \ll M$. *Thus* $X = X \cap (S + L) = S + H$, *where* $H = X \cap L$

(ii) By (3.2).

THEOREM 3.4. *Let M be a module and consider the following conditions:*

- (a) *M is* $ET -$*lifting.*
- (b) *M is* $ET - G^* -$*lifting.*
- (c) *M is* $ET - G^* -$*H-supplemented.*
- (d) *M is* $ET - G^* -$*supplemented.*

Then $(a) \Rightarrow (b) \Longleftrightarrow (c) \Rightarrow (d)$

$proof$: $(a) \Rightarrow (b)$ *is clear*

$(b) \Leftrightarrow (c)$. *This equivalence follows from Theorem 2.3*

$(b) \Leftrightarrow (d)$. This implication follows from the fact that every direct summand is a
supplement.

Proposition(3.5). *If M is a quasi-projective module then the following conditions are equivalent:*

(i) M *is supplemented,*

(ii) M *is* $ET - G^{*}$ *−supplemented,*

References

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For papers published in translation journals, please give the English citation first, followed by the original foreign-language citation [6].

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